Journal
of
The Franklin Institute
Devoted to Science and the Mechanic Arts

Vol. 268 JULY, 1959 No. 1

APPLICATION OF MATRIX METHODS TO THE LINEAR LEAST SQUARES SYNTHESIS OF MULTIVARIABLE SYSTEMS*

BY

R. C. AMARA[1]

ABSTRACT

The class of systems treated in this paper is that characterized by a multiplicity of inputs and outputs. For such linear multivariable systems subject to continuous stochastic inputs with arbitrary self- and cross-correlations, a derivation is made of the equations to be satisfied by the optimum transfer-function matrix. The criterion of performance which is used is the minimization of the sum of the mean-square errors between the set of actual outputs and a set of desired outputs.

The analysis divides the multivariable problem into two principal parts. In the first, designated the filter case, are included those systems in which the only restriction on the linear transfer-function matrix to be realized is that of physical realizability. Included in the second, the control case, are those systems in which a controlled matrix or plant is specified which imposes additional restrictions on the over-all performance of the system.

The results of the analysis are presented compactly in matrix form. For signal power spectra and plant matrices satisfying prescribed conditions, explicit matrix solutions to the minimization equations are obtained.

I. INTRODUCTION

In the field of control engineering a considerable effort has traditionally been devoted to an investigation of systems characterized by a single input and a single output. Until very recently, relatively little attention had been given to the class of systems having a multiplicity of inputs and outputs, designated multivariable systems. In the past few years,[2] however, interest in this area has begun to grow rapidly as evidenced by the increasing number of papers which have appeared on the subject.

* The research reported herein was supported jointly by the U. S. Air Force, Army and Navy under Contract Nonr 225(24) 373 360 with Stanford University.

[1] Senior Research Engineer, Stanford Research Institute, Menlo Park, Calif.

[2] See Bibliography appended to this paper.

(Note—The Franklin Institute is not responsible for the statements and opinions advanced by contributors in the JOURNAL.)

A multivariable system may be defined as a configuration having p input terminals and q output terminals, as shown in Fig. 1. In the work to be described herein, it will be assumed that the multivariable

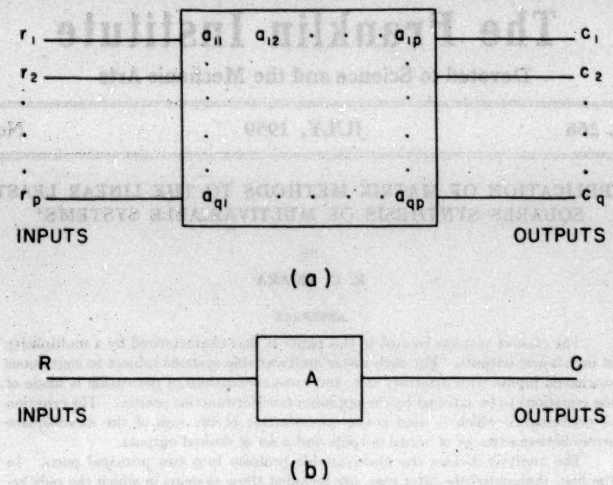


FIG. 1. Representations of a linear multivariable system.

systems under consideration can be described by linear, time invariant, input-output relations over the whole range of operation or over the restricted range of interest. Such assumptions make it possible to describe the action of a system by defining a transfer-function matrix. Thus, using matrix notation and letting

$$R = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_q \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1p} \\ \vdots & \vdots \\ a_{q1} & \cdots \cdots a_{qp} \end{bmatrix},$$

the system input-output relation may be expressed compactly as

$$C = AR, \tag{1}$$

where R, C, and A are the input, output, and transfer-function matrices, respectively. It is seen that a multivariable system derives each output by a simultaneous operation on all its inputs. Furthermore, it is apparent that the class of multivariable systems includes the common

single variable system as a special case. A turbo-jet engine, a steel rolling mill, or a chemical processing plant are but a few of the practical examples of multivariable systems that may be cited.

The purpose of this paper is to demonstrate the application of matrix methods in the synthesis of multivariable systems subject to stationary stochastic inputs. To do this, the following two basic problems are considered:

1. Linear Multivariable Filter System: For a filter subject to random inputs satisfying certain prescribed conditions, solutions are obtained for minimizing each output mean-square error. The only restriction in this case is that of physical realizability.

2. Linear Multivariable Control System: Similarly, for a controlled plant subject to random inputs, solutions are obtained for minimizing the sum of the output mean-square errors. In this case the plant is not available to the designer for adjustment so the synthesis procedure consists of introducing a compensating transfer-function matrix which is physically realizable.

In both cases, the derivation is concerned exclusively with a consideration of the continuous signal case; however, by using similar methods, extensions to hybrid (continuous-discrete) and discrete signal systems can be readily made.

II. FORMULATION OF THE PROBLEM

It is well known that complete freedom is not generally available to the designer of control systems in realizing a particular over-all transfer function. Rather, the elements of the system to be controlled (often designated "fixed" or "plant" elements) impose varying degrees of additional limitations since they are not available to the designer for adjustment. For a multivariable control system the problem might be stated as follows: Given a set of random inputs with arbitrary cross-correlations and a plant matrix which specifies the fixed and unalterable elements to be controlled, what is the transfer function of the compensating matrix that transforms the set of system inputs as closely as possible to a set of desired output functions? It is clear that this class of systems includes the simpler pure filter or "free configuration" class as a special case. Thus, the multivariable problem will be formulated in terms of the control case and a reduction to the filter form will be made at the appropriate time.

The principal properties of the systems to be considered and the criterion which is to be used as a measure of performance are stated below.

1. All signals are members of stationary, random, continuous processes with power spectra expressible as rational functions. In any

practical problem, the usual procedure is to approximate all power spectra by rational functions in order to obtain transfer functions in useful form.

2. The fixed plant matrices as well as the operator matrices to be synthesized are linear, continuous, time invariant, and stable; they may be represented by transfer-function matrices whose elements are expressible as rational functions of complex frequency.

3. The criterion of system performance is the minimization of the sum of the mean-square errors between the set of actual outputs and a set of ideal or desired outputs. This has the merit both of lending itself to analytical treatment and of being applicable to many real problems. The foregoing conditions or assumptions are essentially direct analogues of those normally applying to the equivalent single-variable case.

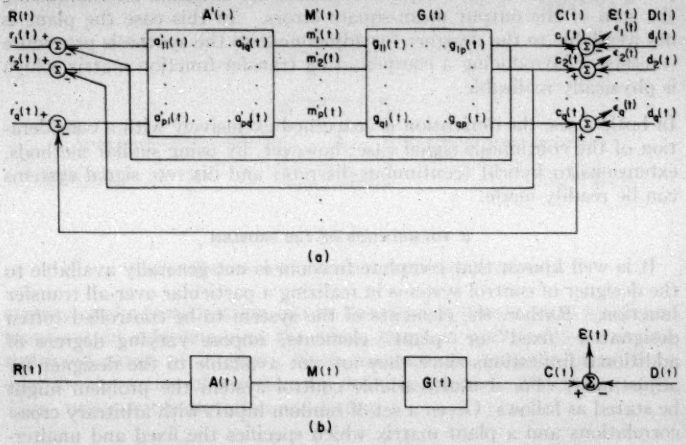


FIG. 2. Feedback and cascade representations of the general control problem.

Figure 2*a* shows the basic multivariable control system; as can be seen, direct negative feedback from output to input is considered to be an integral part of the system. In the figure the plant matrix G is $q \times p$ and the compensating matrix A' is $p \times q$; also, the signals R, M', C, D and $\mathcal{E}$ are $p \times 1$ or $q \times 1$ column matrices as required. Instead of working directly with the feedback configuration shown, it is more convenient to replace it by an equivalent cascade combination of matrices as indicated in Fig. 2*b*. The relationship between the matrices A' and A is readily derived to be

$$A' = A(I - GA)^{-1}. \tag{2}$$

Furthermore, it can be shown that no design freedom is lost in replacing one configuration by the other provided that G is stable, as assumed.

From Fig. 2*b* the total mean-square error $\overline{\delta r^2(t)}$ for a plant with p inputs and q outputs is

$$\overline{\delta r^2(t)} = \sum_{h=1}^{q} \overline{c_h^2(t)} - 2\overline{c_h(t)d_h(t)} + \overline{d_h^2(t)}. \tag{3}$$

Each of the terms within the summation will now be expanded. Working first with $\overline{c_h^2(t)}$, the manipulated variable $m_i(t)$ at the input to the plant is

$$m_i(t) = \sum_{j=1}^{q} \int_{-\infty}^{\infty} d\sigma r_j(t - \sigma)a_{ij}(\sigma), \quad i = 1, 2, \cdots p \tag{4}$$

while the output is

$$c_h(t) = \sum_{i=1}^{p} \int_{-\infty}^{\infty} d\alpha m_i(t - \alpha)g_{hi}(\alpha), \quad h = 1, 2, \cdots q. \tag{5}$$

Substituting Eq. 4 into Eq. 5 and reordering summations and integrations, there is obtained

$$c_h(t) = \sum_{i=1}^{p} \sum_{j=1}^{q} \int_{-\infty}^{\infty} d\alpha g_{hi}(\alpha) \int_{-\infty}^{\infty} d\sigma r_j(t - \sigma - \alpha)a_{ij}(\sigma), \quad h = 1, 2, \cdots q. \tag{6}$$

Using a new set of variables and indices to distinguish the different operations of integration and summation, $c_h(t)$ may also be written as

$$c_h(t) = \sum_{k=1}^{p} \sum_{l=1}^{q} \int_{-\infty}^{\infty} d\beta g_{hk}(\beta) \int_{-\infty}^{\infty} d\rho r_l(t - \rho - \beta)a_{kl}(\rho), \quad h = 1, 2, \cdots q. \tag{7}$$

Now multiplying Eqs. 6 and 7, averaging with respect to time, and noting that

$$\overline{r_j(t - \sigma - \alpha)r_l(t - \rho - \beta)} = \phi_{r_j r_l}(\sigma + \alpha - \rho - \beta) \tag{8}$$

where $\phi_{r_j r_l}(\cdot)$ is the self- or cross-correlation function, there results

$$\overline{c_h^2(t)} = \sum_{i,k=1}^{p} \sum_{j,l=1}^{q} \int_{-\infty}^{\infty} d\alpha g_{hi}(\alpha) \int_{-\infty}^{\infty} d\beta g_{hk}(\beta) \int_{-\infty}^{\infty} d\sigma a_{ij}(\sigma) \times \int_{-\infty}^{\infty} d\rho a_{kl}(\rho)\phi_{r_j r_l}(\sigma + \alpha - \rho - \beta), \quad h = 1, 2, \cdots q. \tag{9}$$

Turning to the second term in Eq. 3 and noting that

$$\overline{r_j(t - \sigma - \alpha)d_h(t)} = \phi_{r_j d_h}(\sigma + \alpha) \tag{10}$$

the result is

$$2\overline{c_h(t)d_h(t)} = 2\sum_{i=1}^{p}\sum_{j=1}^{q}\int_{-\infty}^{\infty} d\alpha g_{hi}(\alpha)\int_{-\infty}^{\infty} d\sigma a_{ij}(\sigma)\phi_{r_j d_h}(\sigma+\alpha), \quad h = 1, 2, \cdots q. \tag{11}$$

The third term of Eq. 3, $\overline{d_h^2(t)}$, will be written in the equivalent form, $\phi_{d_h d_h}(0)$.

Collecting all terms and substituting in Eq. 3 yields

$$\overline{\epsilon_T^2(t)} = \sum_{h=1}^{q}\left\{\sum_{i,k=1}^{p}\sum_{j,l=1}^{q}\int_{-\infty}^{\infty} d\alpha g_{hi}(\alpha)\int_{-\infty}^{\infty} d\beta g_{hk}(\beta)\int_{-\infty}^{\infty} d\sigma a_{ij}(\sigma) \times\int_{-\infty}^{\infty} d\rho a_{kl}(\rho)\phi_{r_j r_l}(\sigma+\alpha-\rho-\beta) - 2\sum_{i=1}^{p}\sum_{j=1}^{q}\int_{-\infty}^{\infty} d\alpha g_{hi}(\alpha) \times\int_{-\infty}^{\infty} d\sigma a_{ij}(\sigma)\phi_{r_j d_h}(\sigma+\alpha)+\phi_{d_h d_h}(0)\right\}. \tag{12}$$

III. DERIVATION OF THE MINIMIZATION EQUATIONS

A minimization will now be performed of Eq. 12 with respect to a_{ij} or a_{kl} by a variational procedure. Let

$$\begin{aligned} a_{ij}(\sigma) &= a_{ij_m}(\sigma) + \epsilon a_{ij_\epsilon}(\sigma) \\ a_{kl}(\rho) &= a_{kl_m}(\rho) + \epsilon a_{kl_\epsilon}(\rho) \end{aligned} \tag{13}$$

where $a_{ij_m}(\sigma)$ or $a_{kl_m}(\rho)$ are the assumed sol. tions and $\epsilon a_{ij_\epsilon}(\sigma)$ or $\epsilon a_{kl_\epsilon}(\rho)$ are the corresponding variations. Substituting the expressions of Eq. 13 into Eq. 12, differentiating with respect to ϵ, setting the value at $\epsilon = 0$ equal to zero and noting that

$$\begin{aligned} \phi_{r_j r_l}(\sigma+\alpha-\rho-\beta) &= \phi_{r_j r_l}(\rho+\beta-\sigma-\alpha), \quad j = l \\ \phi_{r_j r_l}(\sigma+\alpha-\rho-\beta) &= \phi_{r_l r_j}(\rho+\beta-\sigma-\alpha), \quad j \neq l \end{aligned} \tag{14}$$

the set of pq equations may be written in the following form:

$$\sum_{h=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{q}\int_{-\infty}^{\infty} d\sigma a_{ij_\epsilon}(\sigma)\int_{-\infty}^{\infty} d\alpha g_{hi}(\alpha) \times\int_{-\infty}^{\infty} d\beta g_{hk}(\beta)\int_{-\infty}^{\infty} d\rho a_{kl_m}(\rho)\phi_{r_j r_l}(\sigma+\alpha-\rho-\beta) - \sum_{h=1}^{q}\int_{-\infty}^{\infty} d\sigma a_{ij_\epsilon}(\sigma)\int_{-\infty}^{\infty} d\alpha g_{hi}(\alpha)\phi_{r_j d_h}(\sigma+\alpha) = 0, \quad \begin{matrix} i = 1, 2, \cdots p \\ j = 1, 2, \cdots q. \end{matrix} \tag{15}$$

For any i, j the common factor, $\int_{-\infty}^{\infty} d\sigma a_{ij_\epsilon}(\sigma)$, may now be moved outside the summation signs; but, since $a_{ij_\epsilon}(\sigma) = 0$ for $\sigma < 0$, the remaining factor must be zero for $\sigma \geq 0$, so that

$$\sum_{k=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{q}\int_{-\infty}^{\infty} d\alpha g_{ki}(\alpha)\int_{-\infty}^{\infty} d\beta g_{kk}(\beta)\int_{-\infty}^{\infty} d\rho a_{kl}(\rho)\phi_{r_jr_l}(\sigma+\alpha-\rho-\beta) - \sum_{k=1}^{q}\int_{-\infty}^{\infty} d\alpha g_{ki}(\alpha)\phi_{r_jd_k}(\sigma+\alpha) = 0; \quad \sigma \geq 0, \quad \begin{matrix} i = 1, 2, \cdots p \\ j = 1, 2, \cdots q \end{matrix} \tag{16}$$

are the pq minimizing equations which must be satisfied. That the stationary point which has been obtained is in fact a minimum may be verified by examining the sign of $\left.\frac{\partial^2 \overline{\mathcal{E}_T^2(t)}}{\partial \epsilon^2}\right|_{\epsilon=0}$.

When the a_{kl} satisfy Eq. 16 the minimum realizable error, $\overline{\mathcal{E}_T^2(t)}_{\min}$, is achieved. An expression for evaluating it may be obtained directly from Eq. 12 when the conditions of Eq. 16 are satisfied. The result is that

$$\overline{\mathcal{E}_T^2(t)}_{\min} = \sum_{k=1}^{q}\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\phi_{d_kd_k}(0) - \int_{-\infty}^{\infty} d\alpha g_{ki}(\alpha)\int_{-\infty}^{\infty} d\sigma a_{ij}(\sigma)\phi_{r_jd_k}(\sigma+\alpha)\right]. \tag{17}$$

Equations 16 and 17 are more useful as frequency domain expressions. Accordingly, equivalent expressions will be obtained for both. Equation 16 may be considered a function of σ, designated $f_{ji}(\sigma)$; a Fourier transformation results in

$$f_{ji}(s) = \sum_{k=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{q} g_{ki}(-s)g_{kk}(s)a_{kl}(s)\phi_{r_jr_l}(s) - \sum_{k=1}^{q} g_{ki}(-s)\phi_{r_jd_k}(s), \quad \begin{matrix} i = 1, 2, \cdots p \\ j = 1, 2, \cdots q \end{matrix} \tag{18}$$

where $s \triangleq j\omega$, and $f_{ji}(s)$ does not have any poles in the left-half s-plane. Since the factors of the terms within the summations are expressible as rational functions of s, a partial fraction expansion with respect to left-plane poles must equate to zero. Therefore,

$$\left[\sum_{k=1}^{q}\sum_{k=1}^{p}\sum_{l=1}^{q} g_{ki}(-s)g_{kk}(s)a_{kl}(s)\phi_{r_jr_l}(s)\right]^+ = \left[\sum_{k=1}^{q} g_{ki}(-s)\phi_{r_jd_k}(s)\right]^+, \quad \begin{matrix} i = 1, 2, \cdots p \\ j = 1, 2, \cdots q \end{matrix} \tag{19}$$

where $[\cdots]^+$ indicates the expansion in terms of left-plane poles only. Now by using matrix notation the expression may be simplified even further. Let Φ_{rr} be the $q \times q$ matrix of elements $\phi_{r_j r_l}(s)$, Φ_{rd} be the $q \times q$ matrix of elements $\phi_{r_j d_h}(s)$, G be the $q \times p$ matrix of elements $g_{hk}(s)$, and A be the $p \times q$ matrix of elements $a_{kl}(s)$. Equation 19 becomes

$$[\Phi_{rr} A^T G^T \bar{G}]^+ = [\Phi_{rd} \bar{G}]^+, \tag{20}$$

where, for any matrix $X(s)$, X^T is the transpose of $X(s)$ and $X(-s)$ is written as $\bar{X}$. As will be shown in Sec. IV, this equation will prove most useful both in obtaining direct solutions to the control problem and in relating the latter to the solutions obtained for the filter case.

Turning to Eq. 17, it may be written in its transformed version as

$$\overline{\varepsilon_T^2(t)}_{\min} = \sum_{h=1}^{q} \sum_{i=1}^{p} \sum_{j=1}^{q} \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [\phi_{d_h d_h}(s) - g_{hi}(-s) a_{ij}(-s) \phi_{r_j d_h}(s)] ds. \tag{21}$$

Now letting Φ_{dd} be the $q \times q$ matrix of elements $\phi_{d_u d_v}(s)$ and using the other matrices as defined previously, there is obtained

$$\overline{\varepsilon_T^2(t)}_{\min} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \mathrm{tr}\{\Phi_{dd} - \bar{G}\bar{A}\Phi_{rd}\} ds, \tag{22}$$

where $\mathrm{tr}\{\cdots\}$ denotes the trace of a matrix.

IV. EXPLICIT MATRIX SOLUTIONS OF MINIMIZATION EQUATIONS

The set of pq equations represented by Eq. 20 are generalized forms of a modified Wiener-Hopf equation. When the plant matrix G becomes the identity matrix, Eq. 20 reduces to the minimization equation for the filter problem:

$$[\Phi_{rr} A^T]^+ = [\Phi_{rd}]^+. \tag{23}$$

It will be shown that the explicit methods of solution developed for the simpler filter case can be applied directly to the control problem. Therefore, the filter form as represented by Eq. 23 will be considered first. Only explicit matrix methods are included; non-explicit, non-matrix methods of solution are described elsewhere.[3]

A. Filter Problem

An explicit solution to Eq. 23 is possible when Φ_{rr} is diagonal. For this case the method of spectrum factorization developed by Wiener for the single-variable case can be adapted to matrix-valued functions. The formal series of steps becomes:

[3] See (8) in Bibliography.

$$
\begin{aligned}
F &= \Phi_{rr}A^T - \Phi_{rd} \\
\{\Phi_{rr}^-\}^{-1}F &= \Phi_{rr}^+A^T - \{\Phi_{rr}^-\}^{-1}\Phi_{rd} \\
A^T &= \{\Phi_{rr}^+\}^{-1}[\{\Phi_{rr}^-\}^{-1}\Phi_{rd}]^+,
\end{aligned} \tag{24}
$$

where $\Phi_{rr} = \Phi_{rr}^+\Phi_{rr}^-$, and Φ_{rr}^+ has elements with all poles and zeros in the left-half s-plane, while Φ_{rr}^- has elements with all poles and zeros in the right-half s-plane. It should be noted that Eq. 24 includes the single-variable solution as a special case. That is, when Φ_{rd} is also diagonal, A^T is diagonal and it follows that $A^T = A$.

In Eq. 24 the factorization of the matrix Φ_{rr} is possible only because it is assumed to be diagonal. When all cross-power spectra are not zero, no general method for performing this operation is known. For this case, a suitable transformation must be found which can reduce the Φ_{rr} to diagonal form.

The input power spectra matrix Φ_{rr} is Hermitian since $\phi_{r_jr_i}(s) = \phi_{r_ir_j}(-s)$. For such matrices a unitary transformation to a diagonalized form is always possible. This is accomplished by the use of the normalized modal matrix U which is formed by arranging the orthogonalized unit characteristic vectors of Φ_{rr} as its columns. The conversion to diagonal form is made by the matrix operation

$$\hat{\Phi}_{rr} = \bar{U}^T\Phi_{rr}U, \tag{25}$$

where $\hat{\Phi}_{rr}$ is the new diagonal matrix and $\bar{U}^T$ is the transposed conjugate of the modal matrix. The elements of Φ_{rr} are the eigenvalues of Φ_{rr} and are always real. The transformation has the important property that

$$\bar{U}^T = U^{-1} \quad \text{or} \quad \bar{U}^TU = I \tag{26}$$

for which reason it is designated a unitary transformation.

The $q \times q$ error matrix $\Phi_{\epsilon\epsilon}$ may be expressed in terms of the other system matrices as

$$\Phi_{\epsilon\epsilon} = \Phi_{dd} - \bar{A}\Phi_{rd} - A\bar{\Phi}_{rd} + \bar{A}\Phi_{rr}A^T, \tag{27}$$

where all matrices have been previously defined. The minimization problem can now be expressed in terms of the diagonal elements of $\Phi_{\epsilon\epsilon}$ in the following manner:

$$\overline{\mathcal{E}_T^2(t)} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathrm{tr}\{\Phi_{\epsilon\epsilon}\}ds. \tag{28}$$

Thus, it is seen that a minimization of $\overline{\mathcal{E}_T^2(t)}$, the sum of the mean square errors, is in effect a minimization of an integral of the trace of the $\Phi_{\epsilon\epsilon}$ matrix. Another important property of a unitary transformation is

that the trace of a transformed matrix remains invariant. Therefore, if an integral of the trace is minimized in one domain, it is also minimized in the other.

It remains now to establish the conditions under which a useful transformation can be made between the two domains. Although such matrices may be related in any consistent way across the transformation, it has been found that explicit solutions may be obtained only when the correspondence is maintained one-to-one throughout. If the unitary transformation is applied to Eq. 27, the result may be written in the form

$$U^{-1}\Phi_{ss}U = U^{-1}\Phi_{dd}U - U^{-1}\bar{A}UU^{-1}\Phi_{rd}U - U^{-1}AUU^{-1}\bar{\Phi}_{rd}U + U^{-1}\bar{A}UU^{-1}\Phi_{rr}UU^{-1}A^{T}U, \quad (29)$$

where the product $UU^{-1} = I$ has been inserted at appropriate places. By examining Eq. 29 it is possible readily to establish the identity under all conditions of those matrices that appear only once, namely, $\hat{\Phi}_{ss}$, $\hat{\Phi}_{dd}$, and $\hat{\Phi}_{rr}$. For the $\bar{A}$ and $\hat{\Phi}_{rd}$ matrices, it is seen that since

$$\begin{matrix} \bar{\bar{A}} = U^{-1}\bar{A}U \\ \hat{A} = U^{-1}AU \end{matrix} \quad \text{and} \quad \begin{matrix} \bar{\hat{\Phi}}_{rd} = U^{-1}\Phi_{rd}U \\ \hat{\Phi}_{rd} = U^{-1}\Phi_{rd}U \end{matrix} \quad (30)$$

the modal matrix U must be such that $U = \bar{U}$. This in turn means that Φ_{rr} must satisfy the condition that $\Phi_{rr} = \bar{\Phi}_{rr}$; for if

$$\begin{matrix} U^{-1}\Phi_{rr}U = \hat{\Phi}_{rr} \\ \text{and} \\ \bar{U}^{-1}\bar{\Phi}_{rr}\bar{U} = \bar{\hat{\Phi}}_{rr} \end{matrix} \quad (31)$$

then $\Phi_{rr} = \bar{\Phi}_{rr}$ since U and $\hat{\Phi}_{rr}$ are equal to their respective conjugates. A more restrictive condition is imposed on U by the requirement that the elements of A be physically realizable. This is easily seen as follows. Since

$$[\hat{\Phi}_{rr}{}^{+}U^{-1}A^{T}U]^{+} = [\{\hat{\Phi}_{rr}{}^{-}\}^{-1}\hat{\Phi}_{rd}]^{+} \quad (32)$$

an explicit solution for A^{T} can be made only if matrices U^{-1} and U do not have poles in the right-half s-plane. But since $U^{-1} = \bar{U}^{T}$, this means that the elements of U must have no poles. In this case then

$$A^{T} = U\{\hat{\Phi}_{rr}{}^{+}\}^{-1}[\{\hat{\Phi}_{rr}{}^{-}\}^{-1}\hat{\Phi}_{rd}]^{+}U^{-1}. \quad (33)$$

B. Control Problem

Equation 20 may also be written as

$$[\Phi_{rr}A^{T}\Phi_{gg}]^{+} = [\Phi_{rd}\bar{G}]^{+}, \quad (34)$$

where $\Phi_{vv} \triangleq G^T\tilde{G}$. From an examination of Eq. 34 it is clear an explicit solution may be obtained when both Φ_{rr} and Φ_{vv} are diagonal by the methods used in Eq. 24. For example, by reasoning as in Eq. 24 it can be verified that for this case

$$A^T = \{\Phi_{rr}^+\}^{-1}[\{\Phi_{rr}^-\}^{-1}\Phi_{rd}\tilde{G}\{\Phi_{vv}^-\}^{-1}]^+\{\Phi_{vv}^+\}^{-1}. \tag{35}$$

As in Eq. 24, when Φ_{rd} is also diagonal $A^T = A$ and Eq. 35 reduces to the common single-variable case.

A valid multivariable control solution also exists when Φ_{vv}, or G, alone is diagonal. This corresponds to the case in which a number of single-variable plants are fed with inputs which have arbitrary cross-correlations. If G is diagonal, the minimization equation may be written as

$$[\Phi_{rr}A^T\Phi_{vv}^+]^+ = [\Phi_{rd}\tilde{G}\{\Phi_{vv}^-\}^{-1}]^+. \tag{36}$$

But since Φ_{vv}^+ has elements with all its poles in the left-half s-plane, the physical realizability of A'^T defined as

$$A'^T = A^T\Phi_{vv}^+ \tag{37}$$

remains intact. Furthermore, when Φ_{rd}' is defined as

$$\Phi_{rd}' = \Phi_{rd}\tilde{G}\{\Phi_{vv}^-\}^{-1}, \tag{38}$$

Eq. 36 may be written as

$$[\Phi_{rr}A'^T]^+ = [\Phi_{rd}']^+. \tag{39}$$

This is exactly of the form of Eq. 23 with the result that the transformation method described in Sec. IV-A is directly applicable when Φ_{rr} permits.

A direct reduction to an equivalent filter problem may also be possible even when none of the matrices Φ_{rr}, Φ_{rd} or Φ_{vv} is diagonal. For a single-variable control problem, it is well known that design with a minimum-phase plant $g(s)$ is reducible to design for a "free-configuration" filter $a(s)$; in this case, the optimum compensating filter element is determined to be $\frac{a(s)}{g(s)}$. This procedure is valid because the minimum-phase $g(s)$ contains no zeros in the right-half s-plane and therefore performance of the system does not depend on pole-zero cancellations in that part of the plane. For the multivariable problem, it would be expected that a similar reduction is possible.

A minimum-phase plant matrix is defined here as a nonsingular matrix whose determinant has no zeros in the right-half s-plane. If G is minimum phase, then $\{\tilde{G}\}^{-1}$ has elements with all their poles in the right-half s-plane. Therefore, Eq. 20 may be written as

$$F\{\bar{G}\}^{-1} = \Phi_{rr}A^{T}G^{T} - \Phi_{rd}, \tag{40}$$

where the matrix on the left again has elements with no poles in the left-half s-plane and therefore

$$[\Phi_{rr}A^{T}G^{T}]^{+} = [\Phi_{rd}]^{+}. \tag{41}$$

Letting

$$A'^{T} = A^{T}G^{T} \tag{42}$$

with assurance that A'^{T} still has elements with all poles in the left s-plane (since G is stable), then the problem is reduced to solving

$$[\Phi_{rr}A'^{T}]^{+} = [\Phi_{rd}]^{+}. \tag{43}$$

This is again of the form of Eq. 23, the filter-solution minimization equation. If Eq. 43 may be solved by the methods of Sec. IV-A, the solution for A is then obtained by transposing both sides of Eq. 42 so that

$$A = G^{-1}A'. \tag{44}$$

This is in agreement with the expected form of solution since the overall transfer function from input to output then becomes

$$GA = GG^{-1}A' = A' \tag{45}$$

as obtained from Eq. 43.

Up to this point the cases which have been considered either have led to explicit solutions or have been reducible to a form in which the filter method of solution was directly applicable. When Φ_{rr} alone is diagonal, the control problem is reducible to a modified filter form. This represents the case for which a multivariable plant is subject to uncorrelated inputs. For such a system the matrix minimizing relationship may be written in the form

$$[\Phi_{rr}{}^{+}A^{T}\Phi_{gg}]^{+} = [\{\Phi_{rr}{}^{-}\}^{-1}\Phi_{rd}\bar{G}]^{+}. \tag{46}$$

Now let

$$A'^{T} = \Phi_{rr}{}^{+}A^{T}$$

and

$$\Phi_{rd}{}' = \{\Phi_{rr}{}^{-}\}^{-1}\Phi_{rd}\bar{G}, \tag{47}$$

where it is apparent that A'^{T} is physically realizable since $\Phi_{rr}{}^{+}$ has elements with all poles in the left-half s-plane. Substituting the expressions of Eq. 47 into Eq. 46, the result is

$$[A'^{T}\Phi_{gg}]^{+} = [\Phi_{rd}{}']^{+}. \tag{48}$$

Equation 48 is again of the same general form as for the filter problem; the difference is that here Φ_{gg} performs an operation on the columns of A' while in the filter problem Φ_{rr} performs a similar operation on the rows of A. In the latter case an independent solution may be obtained for each row of A; in the former, each column of A' may be independently determined. Thus, for uncorrelated inputs, the compensating matrix is separable on a column basis. Furthermore, it is clear that the explicit solution of Eq. 48 is given by

$$A^T = U\{\hat{\Phi}_{gg}^{+}\}^{-1}[\Phi_{rd}'\{\Phi_{gg}^{-}\}^{-1}]^{+}U^{-1}, \tag{49}$$

when Φ_{gg}, which is Hermitian, satisfies the conditions set forth for Φ_{rr} in Sec. IV-A.

Finally, when the matrices Φ_{rr} or Φ_{gg} do not possess any of the special properties discussed above, Eq. 34 must be solved in its original form. In this form, a reduction to an equivalent filter problem is always possible. To show this it is more suitable to transpose both sides of Eq. 34 to give

$$[\Phi_{gg}^{T}A\Phi_{rr}^{T}]^{+} = [\tilde{G}^{T}\Phi_{rd}^{T}]^{+}. \tag{50}$$

Both Φ_{gg}^{T} and Φ_{rr}^{T} are, of course, still Hermitian. If Φ_{gg}^{T} alone operated on A, the columns of A would have Φ_{gg}^{T} as their coefficient matrix; similarly, if Φ_{rr}^{T} alone operated on A, Φ_{rr} would be the coefficient matrix for the rows of A. Therefore, it is clear that both matrices operating on A simultaneously would produce a composite coefficient matrix which operates on both the rows and columns of A.

A detailed expansion of the operations in Eq. 50 shows that it is always reducible to the form

$$[K\bar{A}]^{+} = [\bar{\Phi}_{rd}]^{+}. \tag{51}$$

In Eq. 51 the matrices $\bar{A}$ and $\bar{\Phi}_{rd}$ are $pq \times 1$ column matrices formed by arranging the elements of A and $\tilde{G}^{T}\Phi_{rd}^{T}$, respectively, sequentially by rows. Also, the equivalent $pq \times pq$ coefficient matrix K is generated from the elements of the matrices Φ_{rr} and Φ_{gg} by the procedure shown in detail in Fig. 3. In this procedure, the operation of matrix superposition, $\{X \diamond Y\}$, is used. This operation is defined as follows: Two matrices of the same dimensions are superposed by multiplying corresponding elements of each matrix to obtain the corresponding element of the new matrix.

In Eq. 51 the matrix K plays the same role in the solution for all the elements of A as Φ_{rr} does in the filter solution for each row of A. Thus, when K satisfies the conditions set forth for Φ_{rr} in Sec. IV-A,

the solution for $\bar{A}$ is given directly by

$$\bar{A} = U\{\hat{K}^{+}\}^{-1}[\{\hat{K}^{-}\}^{-1}\hat{\Phi}_{rd}] + U^{-1}, \tag{52}$$

where U is now the modal matrix for K, and $\hat{K}$ and $\hat{\Phi}_{rd}$ are the transformed versions of K and Φ_{rd}, respectively.

$$\Phi_{qq} = G^{T}\bar{G} = \begin{bmatrix} \phi_{q_1q_1}^{(s)} & \phi_{q_1q_2}^{(s)} & \cdots & \phi_{q_1q_p}^{(s)} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \phi_{q_pq_1}^{(s)} & \cdots & & \phi_{q_pq_p}^{(s)} \end{bmatrix}; \quad \Phi_{rr} = \begin{bmatrix} \phi_{r_1r_1}^{(s)} & \phi_{r_1r_2}^{(s)} & \cdots & \phi_{r_1r_q}^{(s)} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \phi_{r_qr_1}^{(s)} & \cdots & & \phi_{r_qr_q}^{(s)} \end{bmatrix}$$

$$\hat{\Phi}_{q_uq_v} = \overbrace{\begin{bmatrix} \phi_{q_uq_v}^{(s)} & \phi_{q_uq_v}^{(s)} & \cdots & \phi_{q_uq_v}^{(s)} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \phi_{q_uq_v}^{(s)} & \cdots & & \phi_{q_uq_v}^{(s)} \end{bmatrix}}^{q}\Bigg\}q \qquad \begin{matrix} u = 1, 2, \ldots p \\ v = 1, 2, \ldots p \end{matrix}$$

$$\hat{\Phi}_{qq} = \begin{bmatrix} \hat{\Phi}_{q_1q_1} & \hat{\Phi}_{q_2q_1} & \cdots & \hat{\Phi}_{q_pq_1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \hat{\Phi}_{q_1q_p} & \cdots & & \hat{\Phi}_{q_pq_p} \end{bmatrix}; \quad \hat{\Phi}_{rr} = \overbrace{\begin{bmatrix} \Phi_{rr} & \Phi_{rr} & \cdots & \Phi_{rr} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \Phi_{rr} & \cdots & & \Phi_{rr} \end{bmatrix}}^{p}\Bigg\}p$$

$$K = \left\{\hat{\Phi}_{qq} \diamond \hat{\Phi}_{rr}\right\} = \begin{bmatrix} \left\{\hat{\Phi}_{q_1q_1} \diamond \Phi_{rr}\right\} & \left\{\hat{\Phi}_{q_2q_1} \diamond \Phi_{rr}\right\} & \cdots & \left\{\hat{\Phi}_{q_pq_1} \diamond \Phi_{rr}\right\} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \left\{\hat{\Phi}_{q_1q_p} \diamond \Phi_{rr}\right\} & \cdots & & \left\{\hat{\Phi}_{q_pq_p} \diamond \Phi_{rr}\right\} \end{bmatrix}$$

FIG. 3. Generation of the effective coefficient matrix K for the control problem.

APPENDIX

One of the simplest forms of two-input two-output multivariable filter problem will be examined to illustrate the explicit methods of solution. Figure 4 is a diagrammatic representa-

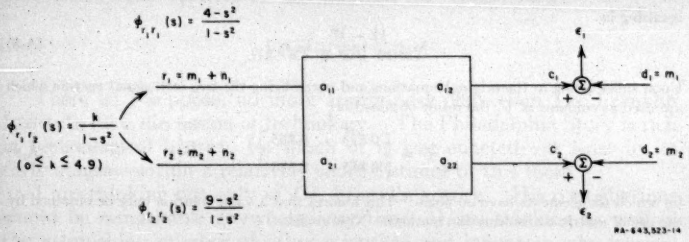


FIG. 4. Block diagram for illustrative example.

tion of the example. Each of the two inputs is composed of a message corrupted by white noise. The noises (n_j) are uncorrelated to each other and to each message, while the messages (m_j) are assumed to possess varying degrees of cross-correlation. The desired or ideal signal at the outputs are the messages themselves. The given data may be summarized as follows:

$$\phi_{r_1r_1}(s) = \phi_{m_1m_1}(s) + \phi_{n_1n_1}(s) = \frac{3}{1-s^2} + 1 = \frac{4-s^2}{1-s^2} \tag{A-1}$$

$$\phi_{r_2r_2}(s) = \phi_{m_2m_2}(s) + \phi_{n_2n_2}(s) = \frac{8}{1-s^2} + 1 = \frac{9-s^2}{1-s^2}$$

$$\phi_{r_1r_2}(s) = \phi_{m_1m_2}(s) = \frac{k}{1-s^2}$$

$$\phi_{r_jd_k}(s) = \phi_{m_jm_k}(s).$$

The permissible range in k may be determined in the following way. Since

$$E[(\lambda_1 m_1 + \lambda_2 m_2 + \cdots \lambda_p m_p)^2] \ge 0 \tag{A-2}$$

then it follows that

$$\sum_{j,l}^{p} \lambda_j \lambda_l E[m_j m_l] \ge 0, \tag{A-3}$$

where $E[\cdots]$ denotes the expected value of the quantity within the brackets. For $p = 2$, it can be verified that this leads to $0 \le k \le 4.9$.

A solution is desired for values of $k = 0$ and $k = 4.9$. For $k = 0$ the explicit single-variable solutions may be obtained by direct substitution in Eq. 24 as follows:

$$A^T = \{\Phi_{rr}^{+}\}^{-1}[\{\Phi_{rr}^{-}\}^{-1}\Phi_{rd}]^{+} \tag{A-4}$$

from which

$$A = \begin{bmatrix} \dfrac{1}{s+2} & 0 \\ 0 & \dfrac{2}{s+3} \end{bmatrix}. \tag{A-5}$$

Using Eq. 22, it follows that

$$\overline{\varepsilon_1^2(t)}_{\min} = 1.0; \quad \overline{\varepsilon_2^2(t)}_{\min} = 2.0. \tag{A-6}$$

For $k = 4.9$ the matrix transformation method will be demonstrated. In testing to determine whether the method may be applied, it is seen immediately that $\Phi_{rr} = \hat{\Phi}_{rr}$. To check that U has elements with no poles, the eigenvalues are obtained from the equation

$$|\Phi_{rr} - \lambda I| = 0 \tag{A-7}$$

resulting in

$$\lambda_1 = \frac{12 - s^2}{1 - s^2}; \quad \lambda_2 = 1. \tag{A-8}$$

Upon substituting in the original equations and normalizing the two orthogonal vectors which are used as columns of U, the result obtained is

$$U = \begin{bmatrix} 0.523 & 0.855 \\ 0.855 & -0.523 \end{bmatrix} \tag{A-9}$$

for which the elements have no poles. This assures that a valid solution may be obtained by the use of matrix transformation method. Now since

$$\hat{\Phi}_{rd} = U^{-1}\Phi_{rd}U = \begin{bmatrix} \dfrac{11.05}{1 - s^2} & 0 \\ 0 & 0 \end{bmatrix} \tag{A-10}$$

and

$$\hat{\Phi}_{rr} = U^{-1}\Phi_{rr}U = \begin{bmatrix} \dfrac{12 - s^2}{1 - s^2} & 0 \\ 0 & 1 \end{bmatrix} \tag{A-11}$$

substitution in Eq. 33 yields

$$A = \frac{1}{s + 3.47}\begin{bmatrix} 0.675 & 1.10 \\ 1.10 & 1.80 \end{bmatrix}. \tag{A-12}$$

The mean-square errors $\overline{\varepsilon_1^2(t)}_{\min}$ and $\overline{\varepsilon_2^2(t)}_{\min}$ are now

$$\overline{\varepsilon_1^2(t)}_{\min} = 0.67; \quad \overline{\varepsilon_2^2(t)}_{\min} = 1.78. \tag{A-13}$$

BIBLIOGRAPHY

(1) A. S. BOKSENBOM AND R. HOOD, "General Algebraic Method Applied to Control Analysis of Complex Engine Types," NACA Tech. Rept. 980, Lewis Flight Propulsion Laboratory, Cleveland, Ohio, 1950.

(2) M. GOLOMB AND E. USDIN, "A Theory of Multidimensional Servo Systems," JOUR. FRANKLIN INST., Vol. 253, pp. 29–57 (1952).

(3) H. FREEMAN, "The Synthesis of Multipole Control Systems," Tech. Rept. T-15/B, Columbia University Electronics Research Laboratories, New York, April 1956.

(4) R. J. KAVANAGH, "The Application of Matrix Methods to Multi-Variable Control Systems," JOUR. FRANKLIN INST., Vol. 262, pp. 349–367 (1956).

(5) N. WIENER AND P. MASANI, "The Theory of Multivariate Stochastic Processes," *Acta Mathematica*, Vol 98, pp. 16–23 (1957).

(6) V. S. PUGACHEV, "The Use of Canonical Expansions of Random Functions in Determining an Optimum Linear System," *Automation and Remote Control* (USSR), Vol. 17, pp. 493–507 (1956).

(7) J. H. WESTCOTT, "Design of Multivariable Optimum Filters," *Trans. A.S.M.E.*, Vol. 80, pp. 463–467 (1958).

(8) R. C. AMARA, "The Linear Least Squares Synthesis of Continuous and Sampled Data Multivariable Systems," Tech. Rept. No. 40, Stanford Electronics Laboratories, Stanford, California, July 1958.

THE GREATEST INVENTION OF THEM ALL*

BY

HENRY B. DU PONT[1]

There is, I suppose, no more appropriate place than The Franklin Institute for a discussion of technology. The Philadelphia Story is rich in technological history, for much of it was enacted—at least in its early stages—within a relatively short distance of this room.

I am thinking not only of Dr. Franklin's work. His contributions would be remarkable anywhere, any time. I am thinking as well of the astonishing number of other scientists and inventors who worked in or near Philadelphia in the early days of the country.

There was the naturalist John Bartram and, a little later, John Audubon whose interest in ornithology began while he was living in Mill Grove, near Mount Carmel. There was Benjamin Rush, the physician. There was David Rittenhouse with his planetarium, and Owen Biddle with his telescope at Cape Henlopen. There was John Fitch, a transplanted Connecticut Yankee, puffing along the Schuylkill and the Delaware in a fire-breathing contraption called a steamboat. A few miles downstream, on Brandywine Creek, there was Oliver Evans who built a flour mill which operated almost without human attention. Among his neighbors was E. I. du Pont, recently arrived from France, who went into the powder business at the urging of Jefferson and made important contributions to the manufacture of explosives then badly needed to clear the forest lands, build the roads and defend the nation.

There was Thomas Paine, remembered as the pamphleteer of the Revolution, but due equal honors for his engineering contributions. One of Paine's more imaginative schemes was an engine driven by a continuous series of tiny gunpowder explosions, actually a sort of internal combustion engine. Unfortunately, the "Internal Explosion Engine" did not work out. I say "unfortunately" because the Du Pont Company was mighty close to bankruptcy for many of its early years, and a large-scale development in this field would have created a welcome new market. However, let us not criticize Citizen Paine for his mistakes. He also developed the iron bridge and had some worthy thoughts about planing machines, cranes, and carriage-wheel construction.

The list of Philadelphia contributors could be extended indefinitely, for, in the Eighteenth and Nineteenth Centuries, this small sector of the

* The 1959 Edward G. Budd Lecture, delivered at the Stated Meeting of The Franklin Institute, Wednesday, April 15, 1959, in the Lecture Hall.

[1] Vice President, E. I. du Pont de Nemours & Company, Wilmington, Delaware.

American geography was the home of dozens who were instrumental to the progress of technology.

It could be argued that "technology" is not the word to use for the scientific contributions of a Franklin or a Bartram, so let me define my terms. Technology is much more than engineering. It is much more than invention, much more than the nuts and bolts, as is sometimes said. Technology, as I see it, is the sum total of man's work in developing the tools and techniques which lend leverage to human effort. Whether a man be a research scientist, or engineer, or an "inspired tinkerer" as some were called, he contributes to and is an integral part of the technology which has shaped this nation.

America in 1800, despite its vast potential wealth, was one of the poorest of the have-not nations, although few nations of that day were well off. The natural resources which we know today were undeveloped, and many of them were unsuspected. In 1800, standards of living even for the well-to-do were crude and, for the mass of the population, they approached the hand-to-mouth level. Farming occupied the attention of more than 90 per cent of the population, and the primitive equipment used could have been recognized by any farmer from the time of Julius Caesar. Manufacture had hardly reached a level where it deserved the title "industry." In 1814, in a report prepared for the United States Senate, a Philadelphian named Tench Coxe estimated that the output of manufacturing in the United States came to a value of something less than $175 million a year. That meant an average of $35 worth of manufactured goods for each man, woman, and child in the population, hardly an impressive amount.

The real poverty of the nation, however, lay in the inadequacy of its human resources to meet growing needs. Tench Coxe was among those who saw that technology held bold possibilities of enriching the nation, and he pleaded for the development of labor-saving machines to lift human burdens and increase output. Men simply could not be spared from the fields. Manpower was so scarce that, in a textile mill, for example, Mr. Coxe said that not more than one employee in eight could be an adult male; the rest of the work had to be done by women and children. Before we send Mr. Coxe to the pillory for suggestions which would shock us today, let me add that no less noble a man than Jefferson shared these views. Coxe was simply stating facts. Men were needed to raise food and could not be spared for manufacture.

In such a restricted economy, there were obviously little money and little time to expend in research and development. The men whose enthusiasm carried them into science or engineering did so, for the most part, on their own initiative, just as we today might take music lessons or write poetry. But, for the necessities of life, they had to turn to other fields. Some of the most significant work done in America came

about as a result of the two-job practice we would now describe as "moonlighting."

Samuel Morse, for example, was a portrait painter by day and a scientist by night. John Fitch sold maps as he traveled the Eastern Seaboard looking for someone, anyone, to invest in his fire-breathing boat. Eli Whitney earned his living as a school master before he earned a nickel from his inventions.

Joseph Henry, another part-time scientist, deplored the fact that his experiments had to be confined to the summer months when the school in which he taught was closed. Evans got along by operating a store, and Paine, who first had plied his trade as a staymaker, later eked out a living by his writing and political work.

Few scientists were as well situated as Franklin whose early successes in publishing enabled him to live amid comforts while indulging his interest in such diverse subjects as the Gulf Stream, electricity, and bifocal spectacles. Most were obliged to expend so much of their energies to earning a livelihood that their most memorable achievements were made on borrowed time. The strain told. Fitch died poor, by his own hand, broken by years of frustration and want. Elias Howe, while enduring hunger and privation, was worn out by toil and died at the age of 48.

To add to the difficulties of these early technologists, poor communication led to frequent duplication of effort, with the result that many things were invented several times by different people who were ignorant of each other's work. When Eli Whitney set up his factory to make guns with interchangeable parts, he spent a large amount of time working out machine tools which already had been developed, or at least anticipated, in other countries. He had no way of knowing that they even existed. Fitch worked for years on a double-acting steam engine, apparently ignorant of the fact that James Watt and Matthew Boulton had already solved many of the problems he faced. Oliver Evans started from scratch, and had no access to technical books, although many had been written in the fields in which he worked. Henry worked for six years on magnetism and electric currents before discovering that Faraday, an ocean away, was doing the same research.

Inevitably, when two or more men brought forth the same discovery at approximately the same time, there were conflicting patent claims. This produced some of the stormiest and most extended legal battles in history. No doubt, it provided amusement for many interested spectators. But it was costly national fun because it dissipated the creative energies of men who could not easily be replaced or duplicated. Genius is too rare a commodity to be squandered lavishly or used inefficiently.

Looking back at the conditions of the Eighteenth and Nineteenth Centuries, it should cause little wonder that technical progress came slowly. The marvel is that anything was accomplished at all, and our

respect and admiration for the talents and persistence of the men involved grow as we come to appreciate their handicaps. They earned every tribute we pay them. I cannot help but wonder how different our history might have been if these extraordinary minds had been unleashed for full-time experiment and invention.

If we were asked to construct a list of the technological developments most fundamental to America's progress, it would be difficult indeed to single out any group upon which we could present a clear case. The nominations most meaningful would be those representative of broad application, spanning all categories of human need.

For, the most notable thing about the technical development of America is that its Industrial Revolution was not confined mainly to manufacturing, as was the case in Europe. Instead, American technical development was spread broadly across the entire economic and scientific field. The growth of our industrial technology was accompanied by parallel gains in agriculture, in communications, and in transportation. There was no single stream of technological advance. Instead, America's progress resulted from the converging of many streams, uniting to produce a flood tide of inventive achievement. In this great creative outpouring, it is difficult to identify any small group of inventions which would be judged outstanding in importance.

Everyone has his own list of "greatest inventions," and every invention has, I suppose, its partisans. My own list has three virtues to commend it. First, it is brief; second, most of my selections are simple; and third, it is based not on scientific preëminence, but on the universal importance to our Economic Revolution.

I would include, for example, the steel plow. A humble device, perhaps—John Deere simply hammered it out of an old circular saw blade. But it speeded the work of the farmer and was the predecessor of a host of other labor-saving farm equipment—the reaper, the binder, the tractor, the combine. Some might argue that these were more significant and, in a way, they would be right. But the important thing about the steel plow is that it was the first in a long line of major advances which lifted man from the subsistence level of agriculture, thus freeing increasing numbers of people for other occupations. Improved farm tools, supplemented by insecticides, weed killers, fertilizers, and better planting and irrigation techniques, have multiplied the efficiency of the farmer by a factor of 25.

Second, I include Oliver Evans' wonderful flour mill, the precursor of automatic control. In terms of scientific merits, it was a modest accomplishment. But it showed the world how a whole series of production operations could be linked together, driven by machinery, and tended by one or two men. All of the technologists who followed Evans, and the millions of consumers who use mass-produced goods, are eternally indebted to this extraordinary man.

The steam locomotive, of course, put power on wheels and gave man a wholly new mobility of special significance in a nation spanning such great distances. The locomotive was not unique to America, and it was more a compendium than an invention, but it should not be ruled out on that score. After all, no nation can claim many major inventions which are all its own, created without outside help. Technology is a cumulative force. To credit all of the people who made the locomotive possible, we would have to track down the inventors of the lever, the wheel, and the inclined plane, not to mention the steam engine and similarly modern developments. Incidentally, some of the most important contributions to railroad development came from this area. George Westinghouse, the inventor of the air brake, was a Pennsylvanian by adoption if not by birth. Matthias Baldwin, the locomotive builder and one of the men who helped found The Franklin Institute, was a Philadelphian. John Edgar Thompson who did as much as any man to build the Pennsylvania Railroad, was born and raised in Delaware County. In the hands of men like these, the railroad reached a high level of development in the United States and, with the vast distances of the continent falling beneath its iron wheels, the locomotive counts more heavily in our history than many inventions we like to think of as 100 per cent home grown.

Fourth on the list is the telegraph which presaged the telephone and did more to speed communication than any device since the printing press and movable type.

I have mentioned here four basic inventions covering each of the major fields. There is a fifth which pertains to all. I think, in any list of inventions, we should include this one which, strictly speaking, is not an invention at all, but a legal fiction. Yet it seems to me to equal in importance any specific invention in history. This is the development of the modern corporation.

The industrial corporation, as we know it today, is a creature of technology just as certainly as the flying machine or the cotton gin, and it has proved the most effective device for extending technology which man has ever known. Its contributions have been unique.

For the modern industrial establishment is the agency which made the pursuit of technology a full-time job, and not for a few men, but for hundreds of thousands. Industry employs today more than 550,000 scientists and engineers, a majority of the nation's technical work force. Such figures amaze few today. But, in contrast to the struggling little band of amateurs of the early days, it is a vast army.

Development of the industrial organization, as we now recognize it, came too late to aid the earlier inventors. In some ways, the "corporate" form of organization dates from medieval times, but it was not until the Twentieth Century that the corporation came into wide usage as the structural form of large-scale organization. In this cen-

tury, it has changed the world. It has provided a focal point for the development of technology, taking it out of the sideline class and making it a full-time profession. The corporation has brought to the support of technology an extraordinary number and variety of services and resources. In our own company, as an example, we have about 2400 technically trained employees in research and development, assisted by more than 3500 specialists and technicians. Du Pont spent $90 million last year to underwrite their work.

All the facilities which can be mustered are employed to assist the research worker and make his effort more productive. No longer does he work in ignorance of what others are doing: Today, few developments throughout the world escape notice, for there is a constant interchange of information, and scientists have major reference publications at their finger tips. Our own technical libraries stock over 800 different periodicals, and we have one divisional group whose sole function is to keep track of research discoveries elsewhere and pass on pertinent information. Whether new ideas originate in our own laboratories or elsewhere, they are eagerly studied and their applicability is quickly appraised. If one man's experience is insufficient to evaluate the work, another's frequently will prove admirable, offering, as it does, a different background and perspective.

What is true in Du Pont is true in many other companies today. Large corporations consolidate information from many disciplines of science and engineering and bring to scientific development a breadth of vision extending far beyond any one man's capacities. The individual scientist is stimulated and encouraged. His discoveries no longer are left to languish on the laboratory shelf while their creator goes out to search for a sponsor.

In addition, the development of the modern corporation has created the national resources which support technology in many other types of institutions. The research efforts of universities, private foundations, and government laboratories contribute importantly to the advancement of American science and engineering, but these institutions could not exist were it not for the wealth generated by American industry.

The size of the institutions furthering technology has become an important element in our progress. With its engineering and production resources, the modern industrial organization can direct large groups of technologists—people skilled in many areas of science—to projects no one individual technologist could undertake. For example, it is possible to produce penicillin in a one-man laboratory in quantities sufficient to save a few lives. Produced in major pharmaceutical plants, penicillin has saved the lives of millions. I remember from my early interest in flying that a number of enterprising individuals built airplanes in backyard garages. Perhaps some still do. They would be the first

to admit, however, that their efforts, in scale or in volume, are not quite comparable to the accomplishments of North American or Boeing.

Some of my colleagues at Du Pont, given a few days' notice and a small set of laboratory apparatus, could formulate a kind of nylon and spin a bobbin or two of yarn. But, to supply a national market, this would be like trying to feed an army with the grain you could raise in a flower pot. Moreover, the factor of size also has an important value to national defense. During World War II, for example, the government asked Du Pont to build the atomic materials plant at Hanford, Washington. More recently, we were asked to undertake the construction of the Savannah River hydrogen-materials plant. These were perhaps the most ambitious engineering projects in history; they could not have been completed without the services of a large institution employing thousands of specialists in dozens of fields.

Quite obviously, all of the technological efforts of companies like Du Pont are not of this magnitude. And, obviously, not all of the technological jobs required for the nation's advancement require large institutions. Some of the most significant accomplishments have come, and will continue to come, from smaller organizations. In technology, as in other facets of American life, there is a need for business units of every size. Small firms supply companies like Du Pont with tools, materials, and special services which are basic to large-scale research and production efforts. At the same time, the small are important customers of the large, buying products which only the large firms can produce in a volume sufficient to meet national needs. Very often, the small firm makes its contribution to technology and finds its route to commercial success by developing and marketing a product created in the research laboratories of a large company. Thus, small and large business units work as a team to expand America's technology, with each contributing its special talents.

Four years ago, the Swiss economist, William Rappard, set down his views on the causes of America's prosperity. Most important of all, he wrote, were the application of science to production, the "passion" for efficiency in industry, mass production to bring large quantities of goods to people at low cost, and the spirit of competition. It is worth noting that the development of large industrial units is a factor in each of Professor Rappard's considerations. These organizations, consolidating many talents, have provided to technology and the technologist a whole new dimension of performance and potential. They have elevated the horizons of science and engineering, and they have literally extended man's reach into the boundless areas of outer space.

When the history of our era is written, the birth and development of the modern corporation must be recorded as a vital factor in our technological progress. Indeed, the corporation may well prove to be the greatest invention of them all.

OBLIQUE PERFORATION OF A THIN PLATE BY A TRUNCATED CONICAL PROJECTILE*

BY

MELVIN ZAID[1] AND BURTON PAUL[2]

ABSTRACT

The authors' previous analyses of normal perforation of thin plate by high speed projectiles are generalized to include truncated cones, at oblique striking angles. A complete velocity-displacement history is obtained for the most general case. Results are presented in graphical form.

I. NOMENCLATURE

$A(\delta^{**})$ = area swept out by meplat in plane of plate
a_1 = radius of projectile meplat
a_3 = radius of projectile body
a, δ = polar coordinates in plane of plate
a', δ = polar coordinates relating to slug in plane of plate
a'', δ = polar coordinates relating to projectile body in plane of plate
h_o = thickness of target plate
$I_1, I_2, \cdots I_6$ = integrals
K = experimental coefficient
l_1 = distance from cone apex to meplat
l_3 = distance from cone apex to projectile body
$M(x)$ = effective mass of target plate
m = mass of projectile
r_0 = initial radius of generic element of target plate
t = time
v = instantaneous projectile velocity
v_s = striking velocity of projectile
Δv = velocity lost by projectile during penetration
W = weight of projectile
x, y, z = coordinate axes related to projectile
x', y', z = rotated coordinate axes related to plate
α = semi-vertex angle of projectile cone
γ = weight density of target plate
δ^* = angle associated with reverse petal formation

* The original results in this paper were obtained in the course of research sponsored by Pitman Dunn Laboratories, Frankford Arsenal, under Contract DAI-36-038-507-ORD-(P)-339 and published in progress reports 1 to 8 (Project HEAP) by Bulova Research and Development Laboratories, Inc., Woodside, N. Y.

[1] Technical Consultant, Plainview, N. Y.

[2] Mechanical Engineering Department, Brown University, Providence, R. I.

δ^{**} = angle associated with slug punching
δ^{***} = angle associated with petal separation
ξ = axial displacement of generic element of target plate[3]
ρ = mass density of target plate
ϕ = angle of obliquity

II. INTRODUCTION

In previous papers (1, 2, 3, 4)[4] a general method for the analysis of projectile penetration at normal incidence was presented. This analysis, based on a momentum technique, is valid for both thin and thick targets. The practical application of this technique requires that a realistic model of the kinematic process be visualized. Several such models have been proposed for thin targets and one that showed good correlation between experimental and theoretical results has been examined in detail. Recognizing that inertia effects predominate over material strength, at sufficiently high velocities, the target plate is conceived as an inert sheet with no resistance to bending. It seems desirable to extend this model to the more complex phenomenon of oblique penetration of thin plates by a truncated conical projectile. As would be expected, various results for oblique and normal perforation by both the conical and truncated conical projectile can be obtained from the general solution derived in this paper.

III. ASSUMPTIONS

Although the present theory allows for the plate material to stretch circumferentially in order to conform to the projectile surface, it has been observed experimentally that such stretching does not occur. Instead, radial cracks appear which divide the target into a number of sectors, called petals, surrounding the center of impact. Although the following analysis is valid whether or not such behavior occurs, the inclusion of the concept of petalling aids in an intuitive understanding of the perforation process.

The kinematic model to be investigated is characterized completely by the following four assumptions:

1. All target elements conform to the surface of the projectile so long as the force between the projectile and the particular element is compressive.
2. All particles in the petal are assumed to move parallel to the generators of the cone.
3. No significant radial stretching of the petals occurs.
4. The projectile velocity vector remains invariant during the entire perforation process.

[3] The subscripts f, r refer to forward and rearward motion, respectively.

[4] The boldface numbers in parentheses refer to the references appended to this paper.

The second statement in assumption (1) follows from the fact that the direction of the compressive force between the projectile and petal element is essentially invariant, so that Newton's law becomes

$$\frac{d}{dt}(\text{Petal momentum}) = F \tag{1}$$

in the direction of the force. Since this force F must be non-negative at all times during the contact, it follows that the petal element momentum is either constant or monotonically increasing in the direction of the force vector. In the event that a geometric constraint inherent in the model requires a decrease in the numerical value of the momentum in the petal element, it will be assumed that the element separates from the projectile with its maximum momentum and remains separated. The actual phenomenon that this separation replaces is much more complicated, since it involves petal rotation combined with a partial contact between the projectile and petal. In addition, it is conceivable that because of strength effects, a separated petal can re-establish contact with the projectile. In this investigation, these added effects will be considered negligible.

An alternate way of stating the first assumption is that the projectile velocity is sufficiently great so that plastic and elastic waves cannot escape radially from the projectile ogive. In the case under consideration, since the deformation of the petal elements is much greater than the elastic deformation in the immediate region of the projectile, it is adequate to take the projectile velocity sufficiently large so that the plastic waves cannot escape radially from the projectile cone.

Assumption (1) leads to the conclusion that the meplat[5] will punch out all the undeformed plate in its path. This result has been discussed earlier (4), and will be referrred to as "slug punching."

Assumption (1) leads to the ambiguity that any petal element can take two possible positions (rearward or forward) on the projectile surface, as shown in Fig. 1. Since, as will be shown, the petal element translates parallel to itself at constant velocity, the only force (F_1

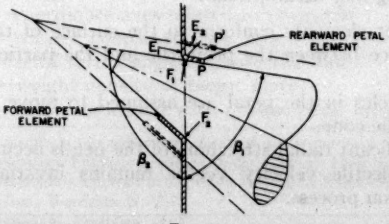


FIG. 1.

[5] The meplat is the flat surface on the projectile nose.

or F_2) on the element is that which is required to change the velocity of the incremental mass (E_1) from zero to the velocity of the petal already in motion (E_2). Hence this force must act at the root, as shown in Fig. 1. From this it is concluded that the petal will initially move and continue in the same direction as the force F_1 or F_2. The forces F_1 and F_2 are made up of two components: one normal to the projectile surface and the other a friction component, parallel to the surface. It has been shown (5) that the coefficient of friction is very small because of local melting, so that the forces F_1, F_2 are essentially normal to the projectile profile. Thus, neglecting friction, the petal will take the direction that has a minimum angle of rotation β from the undeformed to final position. This is also the deformation pattern by which the plate absorbs a minimum amount of bending energy. The energy lost due to crack formation is essentially independent of β, and because of these cracks the strain energy due to circumferential stretching is essentially zero. Thus a minimum angle of rotation β corresponds to a minimum amount of total energy absorbed by the plate. This seems to be an intuitively reasonable deformation mode and will be used throughout this section.

Such forward and backward petalling as shown in Fig. 1 has been observed in many actual firings. At the present time, however, there is no experimental evidence as to the exact value of β for which backward or forward rotation just occurs.

Assumption (2) is made for reasons of mathematical simplicity.

Assumption (3) has been verified in several cases by rolling back the petals of the deformed plate after perforation.

Assumption (4) is not exactly true but has been made for reasons of mathematical simplicity. In the case of thin plates, it is expected to be quite accurate and, if required, it can be refined by calculating the force-time history on the projectile during perforation, and deducing a new projectile path and velocity-time relationship. By such an iterative procedure the more correct projectile path, forces, velocities, etc., can finally be obtained.

IV. CLASSIFICATION OF PERFORATION PROCESS

a. Impact Obliquity

High obliquity perforation is characterized by the existence of some rearward petal elements. Low obliquity impact has no rearward petal elements. By the proper consideration of β associated with assumption (1), the limits for each type of impact can be expressed mathematically as

Low obliquity: $$0 \leq \alpha - \phi \leq \pi/2 - 2\phi \tag{2}$$

High obliquity: $$0 \leq \phi - \alpha \leq \pi/2 - 2\alpha \tag{3}$$

Very high obliquity: $$\pi/2 \leq \phi + \alpha. \tag{4}$$

A graphical presentation which aids in the physical understanding of these limits is shown in Fig. 2. The region of very high obliquity

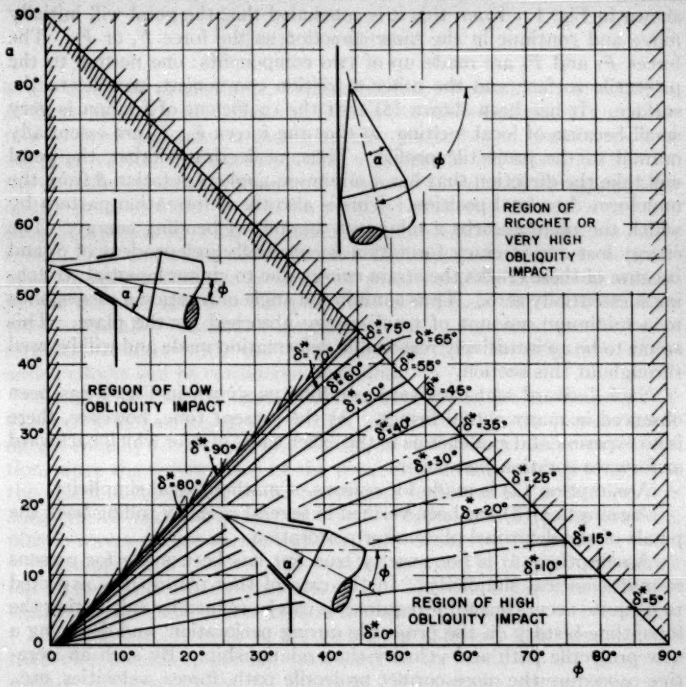


FIG. 2. Type of obliquity classification.

will not be discussed in this paper. The lines of constant δ^* indicated in the region of high obliquity will be discussed in the section on petal formation.

b. Degree of Perforation

The following classification for degree of perforation is illustrated in Fig. 3.

Phase I: slug formation starts, petal elements begin to form.
Phase II: slug has been completely formed, petals conform to conical profile around entire periphery.

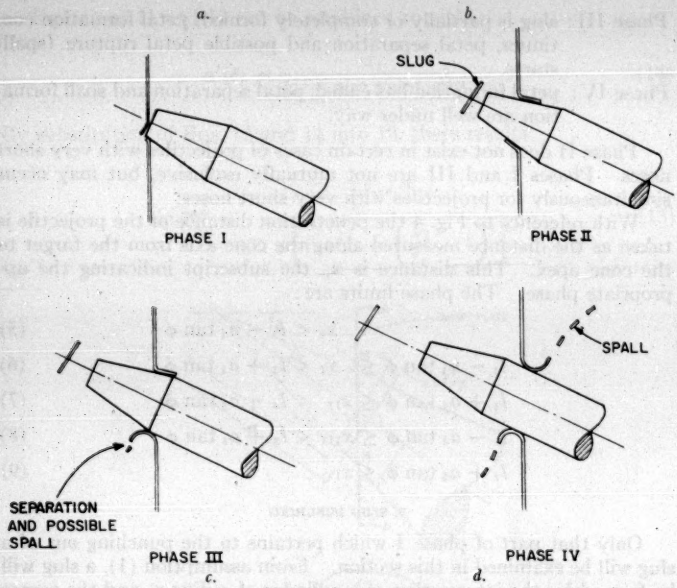

a.
b.
SLUG
PHASE I
PHASE II
SPALL
SEPARATION
AND POSSIBLE
SPALL
PHASE III
PHASE IV
c.
d.

FIG. 3.

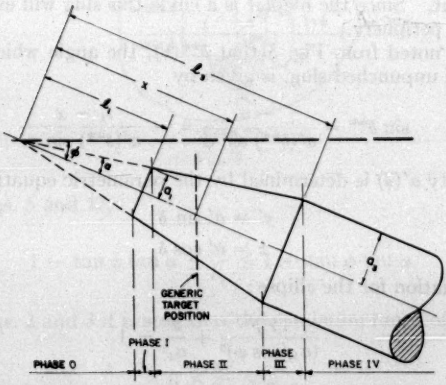

GENERIC
TARGET
POSITION
PHASE I
PHASE 0
PHASE II
PHASE III
PHASE IV

FIG. 4.

Phase III: slug is partially or completely formed, petal formation continues, petal separation and possible petal rupture (spall) starts.

Phase IV: petal formation has ended, petal separation and spall formation are well under way.

Phase II does not exist in certain cases of projectiles with very short noses. Phases I and III are not mutually exclusive, but may occur simultaneously for projectiles with very short noses.

With reference to Fig. 4 the penetration distance of the projectile is taken as the distance measured along the cone axis from the target to the cone apex. This distance is x_i, the subscript indicating the appropriate phase. The phase limits are:

$$x_0 < l_1 - a_1 \tan \phi \tag{5}$$

$$l_1 - a_1 \tan \phi \leq x_I < l_1 + a_1 \tan \phi \tag{6}$$

$$l_1 + a_1 \tan \phi \leq x_{II} < l_3 - a_3 \tan \phi \tag{7}$$

$$l_3 - a_3 \tan \phi \leq x_{III} < l_3 + a_3 \tan \phi \tag{8}$$

$$l_3 + a_3 \tan \phi \leq x_{IV}. \tag{9}$$

V. SLUG PUNCHING

Only that part of phase I which pertains to the punching out of a slug will be examined in this section. From assumption (1), a slug will be formed by the intersection of a cylinder of radius a_1 and the target plate. At any projectile position x, the swept-out area $A(\delta^{**})$ will be a segment of an ellipse. At $x = x_{I\max}$ the complete ellipse will have been punched out. Since the meplat is a circle this slug will extend beyond the meplat periphery.

a. It is noted from Fig. 5 that $\delta^{**}(x)$, the angle which marks the start of the unpunched slug, is given by

$$\sin \delta^{**} = \frac{-u}{a'(\delta^{**}) \sin \phi} = -\frac{l_1 - x}{a'(\delta^{**}) \sin \phi}. \tag{10}$$

The quantity $a'(\delta)$ is determined by the parametric equations:

$$\begin{aligned} y' &= a' \sin \delta \\ z &= a' \cos \delta \end{aligned} \tag{11}$$

and the equation for the ellipse:

$$\frac{(y')^2}{(a_1/\cos \phi)^2} + \frac{z^2}{a_1^2} = 1 \tag{12}$$

where

$$a_1 = l_1 \tan \alpha. \tag{13}$$

Equations 11 and 12 can be combined to yield:

$$a'(\delta) = \frac{a_1}{(1 - \sin^2 \delta \sin^2 \phi)^{1/2}}. \tag{14}$$

By substitution of Eqs. 13 and 14 into 10, there results

$$\sin \delta^{**} = \frac{\dfrac{x_I}{l_1} - 1}{\sin \phi \left[\tan^2 \alpha + \left(\dfrac{x_I}{l_1} - 1 \right)^2 \right]^{1/2}}. \tag{15}$$

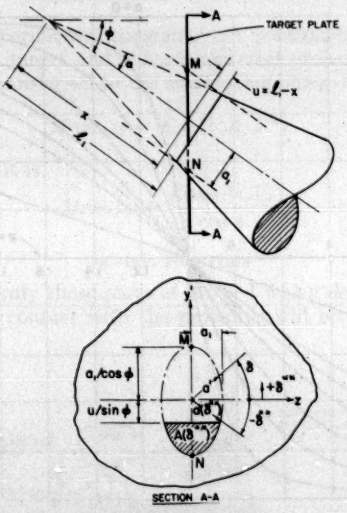


FIG. 5.

From Eqs. 5 and 13,

$$1 - \tan \phi \tan \alpha \leq \frac{x_I}{l_1} \leq 1 + \tan \phi \tan \alpha. \tag{16}$$

From Eqs. 2 and 3 it is seen that the maximum range of x_1/l_1 is

$$0 \leq \frac{x_I}{l_1} \leq 2 \tag{17}$$

The variation of δ^{**} is

$$-\pi/2 \leq \delta^{**} \leq \pi/2 \tag{18}$$

The termination of phase I is indicated by $\delta^{**} = \pi/2$. Figure 6 is a graph of $\sin\phi \sin\delta^{**}$ over the entire range of x_1/l_1.

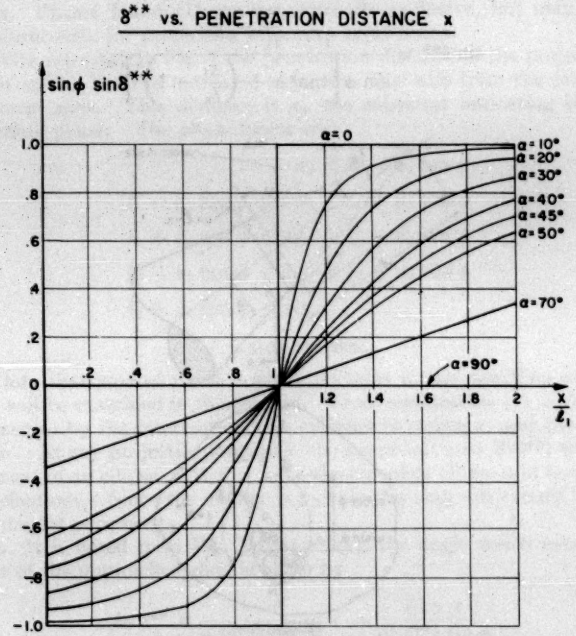


FIG. 6. δ^{**} *versus* penetration distance x.

b. The area $A(\delta^{**})$ is given by

$$A(\delta^{**}) = 2\left[\int_{-\pi/2}^{\delta^{**}} \tfrac{1}{2}a'^2 d\delta - \tfrac{1}{2}a'^2(\delta^{**}) \sin(-)\delta^{**} \cos(-)\delta^{**}\right]. \tag{19}$$

Using Eq. 14 in Eq. 19 results in

$$A(\delta^{**}) = a_1^2\left[\frac{\pi/2 + \tan^{-1}(\cos\phi \tan\delta^{**})}{\cos\phi} + \frac{\sin\delta^{**}\cos\delta^{**}}{1 - \sin^2\phi \sin^2\delta^{**}}\right]. \tag{20}$$

When $\delta^{**} = \pi/2$ at the end of phase I,

$$A\left(\frac{\pi}{2}\right) = \frac{\pi a_1^2}{\cos\phi} \tag{21}$$

which is the area of the ellipse.

c. The mass of the slug in motion at any position x_I is given by

$$M_{slug} = \rho h_o A(\delta^{**}). \tag{22}$$

As discussed in a previous paper (**4**), the velocity of this slug is taken as

$$V_{slug} = KV_s, \quad 1 \leq K \leq 2 \tag{23}$$

where K is an experimental constant. (A reasonable value of K for T 2024 aluminum target and a hardened steel projectile is $K = 1.2$.) The momentum transferred to the slug during phase I is

$$Mom_{slug} = K\rho h_o V_s A(\delta^{**}) \tag{24}$$

and for $x \geq x_{I\max}$ it is

$$Mom_{slug} = \frac{K\rho\pi a_1^2 h_o V_s}{\cos\phi}. \tag{25}$$

VI. PETAL FORMATION

Phase II and only those parts of phase I which deal with the petal while in intimate contact with the projectile will be examined in this section.

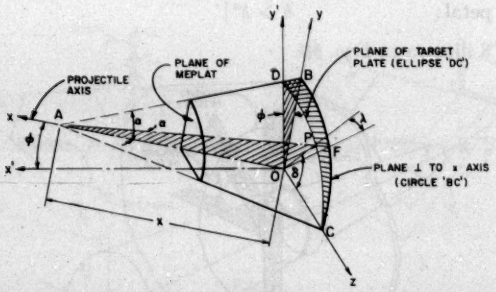


FIG. 7.

Figure 7 is a three-dimensional drawing of one quadrant of the truncated cone. The quarter circle BC is formed by the intersection of the cone and the plane normal to the cone axis. The quarter ellipse

DC is formed by the intersection of the target and the cone. $\overline{AP}$ connects the cone vertex and the generic point P on the ellipse DC. The plane AOP passes through the cone axis and the line $\overline{AP}$.

a. With reference to Fig. 7, following the reasoning that leads to Eqs. 2, 3 and 4, the conditions for forward or rearward petal movement are

$$\left.\begin{array}{ll}\text{Forward petal:} & \alpha - \lambda > 0 \\ \text{Rearward petal:} & \alpha - \lambda < 0\end{array}\right\}. \tag{26}$$

Thus the angle λ^* that separates the regions of forward and rearward petalling is given by

$$\lambda^* = \alpha. \tag{27}$$

By consideration of the geometry associated with Fig. 7, the following relationship is obtained:

$$\sin \delta = \frac{\sin \lambda}{\sin \phi}. \tag{28}$$

Using Eq. 27 in 28,

$$\sin \delta^* = \frac{\sin \alpha}{\sin \phi}, \quad \phi > \alpha. \tag{29}$$

The plot of $\delta^*(\alpha, \phi)$ is shown in Fig. 2. It is noted that $0 \leq \delta^* \leq \pi/2$ in all cases. For $\phi = \alpha$, $\delta^* = \pi/2$. For $\phi < \alpha$, a low obliquity condition, Eq. 29 does not hold and is replaced by $\delta^* = \pi/2$.

The forward, rearward petal criterion becomes:

$$\left.\begin{array}{ll}\text{Forward petal:} & \delta < \delta^* \\ \text{Rearward petal:} & \delta > \delta^*\end{array}\right\}. \tag{30}$$

Figure 8 illustrates Eqs. 30.

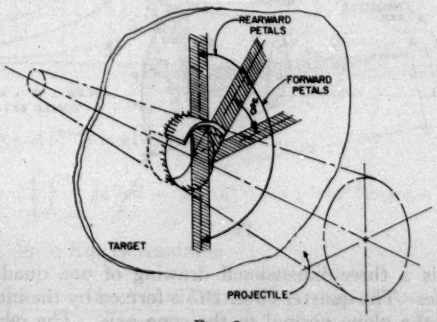


FIG. 8.

b. With reference to Fig. 7, the radius a in the plane of the target is obtained from

$$\frac{\overline{OP}}{\sin \angle OFP} = \frac{\overline{OF}}{\sin \angle OPF} \tag{31}$$

where

$$\begin{aligned} \overline{OP} &= a \\ \overline{OF} &= x \tan \alpha \\ \angle OFP &= \frac{\pi}{2} - \alpha \\ \angle OPF &= \frac{\pi}{2} + \alpha - \lambda. \end{aligned} \tag{32}$$

By substitution of Eqs. 28 and 32 in Eq. 31, there results

$$a = \frac{x \tan \alpha}{(1 - \sin^2 \phi \sin^2 \delta)^{1/2} + \tan \alpha \sin \phi \sin \delta}. \tag{33}$$

With reference to Fig. 9, the incremental petal element mass dM is

$$dM = \rho h_o dA = \rho h_o \left(\frac{a^2 - a'^2}{2} \right) d\delta, \tag{34}$$

where the petal element is defined as the material included by the angle $d\delta$ on the deformed target plate.

c. To obtain the velocity ξ of a generic particle in the deformed target, assumptions (1), (2) and (3) are used. The deformation pattern is then as shown in Fig. 9. These are cross sections showing the

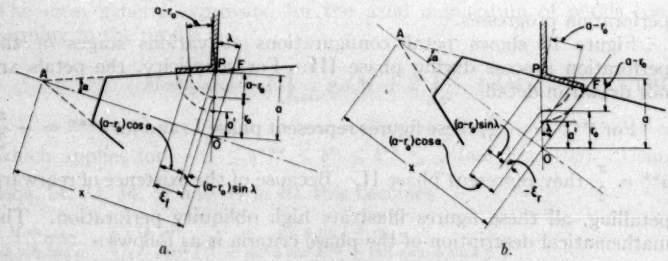


FIG. 9.

forward and rearward petal movement in the plane AOP.

From Fig. 9*a*, the forward particle displacement is

$$\xi_f = (a - r_0)(\cos \alpha + \sin \lambda). \tag{35}$$

Following the motion of a generic particle: $\dot{r}_0 = 0$, and by assumption (2) and Eq. 28, $\dot{\lambda} = 0$. Thus

$$\dot{\xi}_f = \dot{a}(\cos\alpha + \sin\lambda) = v(\cos\alpha + \sin\lambda)\frac{da}{dx}. \tag{36}$$

Substitution of Eqs. 28 and 33 into 36 leads to

$$\dot{\xi}_f = v\frac{\tan\alpha\,(\sin\alpha\sin\delta + \cos\alpha)}{(1 - \sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta}. \tag{37}$$

From Fig. 9*b* the rearward particle displacement is

$$-\xi_r = (a - r_0)(\cos\alpha - \sin\lambda) \tag{38}$$

and, by the same considerations as for the forward particle,

$$\dot{\xi}_r = v\frac{\tan\alpha\,(\sin\phi\sin\delta - \cos\alpha)}{(1 - \sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta}. \tag{39}$$

It is noted that the axial particle velocities are independent of r_0, and hence are constant along any cone generator. Since the petal forms along a straight line, it follows that it must move with constant velocity.

d. To evaluate the petal momentum, it is necessary to account for several possible phenomena. In order to expedite this task a new angle δ^{***} will be defined. δ^{***} is the value of δ which marks the start of the region of petal element separation from the projectile. This value will be calculated in the section on petal separation, where it is shown that separation always starts at $\delta^{***} = -\frac{\pi}{2}$, and increases as perforation progresses.

Figure 10 shows petal configurations at various stages of the perforation process during phase III. For simplicity, the petals are not drawn in detail

For $\delta^{***} = -\frac{\pi}{2}$, these figures represent phase I; and for $\delta^{***} = -\frac{\pi}{2}$, $\delta^{**} = \frac{\pi}{2}$, they represent phase II. Because of the existence of rearward petalling, all these figures illustrate high obliquity perforation. The mathematical description of the phase criteria is as follows:

Phase I: $\quad -\frac{\pi}{2} < \delta^{**} < \frac{\pi}{2}, \quad \delta^{***} = -\frac{\pi}{2}$

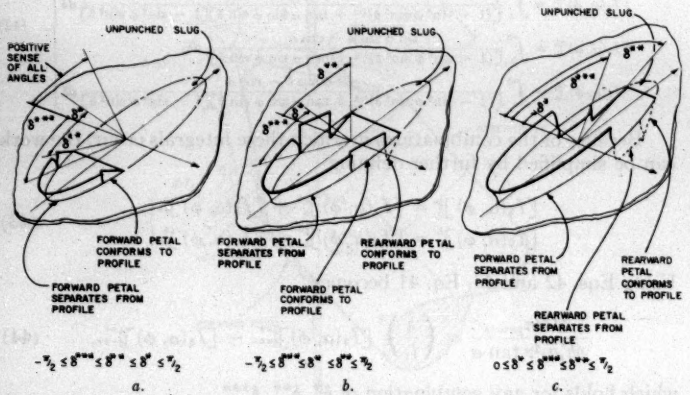
Phase II: $\quad \delta^{**} = \frac{\pi}{2}, \qquad \delta^{***} = -\frac{\pi}{2}$

Phase III: $-\frac{\pi}{2} < \delta^{**} \leq \frac{\pi}{2}, \quad -\frac{\pi}{2} < \delta^{***} < \frac{\pi}{2}$

Phase IV: $\delta^{**} = \delta^{***} = \frac{\pi}{2}$

Low Obliquity: $\delta^{*} = \frac{\pi}{2}$

High Obliquity: $0 < \delta^{*} < \frac{\pi}{2}$


UNPUNCHED SLUG
POSITIVE SENSE OF ALL ANGLES
FORWARD PETAL CONFORMS TO PROFILE
FORWARD PETAL SEPARATES FROM PROFILE
UNPUNCHED SLUG
FORWARD PETAL SEPARATES FROM PROFILE
REARWARD PETAL CONFORMS TO PROFILE
FORWARD PETAL CONFORMS TO PROFILE
UNPUNCHED SLUG
FORWARD PETAL SEPARATES FROM PROFILE
REARWARD PETAL CONFORMS TO PROFILE
REARWARD PETAL SEPARATES FROM PROFILE

$-\pi/2 \leq \delta^{***} \leq \delta^{**} \leq \delta^{*} \leq \pi/2$ $\quad -\pi/2 \leq \delta^{***} \leq \delta^{**} \leq \delta^{*} \leq \pi/2$ $\quad 0 \leq \delta^{*} \leq \delta^{***} \leq \delta^{**} \leq \pi/2$

a. *b.* *c.*

FIG. 10.

The most general expression for the axial momentum of petals conforming to the profile is

$$Mom_{\text{petals}} = 2\int_{\delta^{***}}^{\delta^{*}} \xi_f dM + 2\int_{\delta^{*}}^{\delta^{**}} \xi_r dM \tag{40}$$

which applies for $-\frac{\pi}{2} \leq \delta^{***} \leq \delta^{*} \leq \delta^{**} \leq \frac{\pi}{2}$ (see Fig. 10*b*). Using Eqs. 14, 33, 34, 37 and 39 in 40, this becomes

$$\begin{aligned}\frac{Mom_{\text{petals}}}{\rho h_0 a_1^2 v \tan\alpha} = {} & \left(\frac{x}{l_1}\right)^2 \int_{\delta^{***}}^{\delta^{*}} \frac{\sin\phi\sin\delta + \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta]^3}\, d\delta \\ & - \int_{\delta^{***}}^{\delta^{*}} \frac{\sin\phi\sin\delta + \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta][1-\sin^2\phi\sin^2\delta]}\, d\delta \\ & + \left(\frac{x}{l_1}\right)^2 \int_{\delta^{*}}^{\delta^{**}} \frac{\sin\phi\sin\delta - \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta]^3}\, d\delta \\ & - \int_{\delta^{*}}^{\delta^{**}} \frac{\sin\phi\sin\delta - \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta][1-\sin^2\phi\sin^2\delta]}\, d\delta. \end{aligned} \tag{41}$$

For the case of $-\frac{\pi}{2} \leq \delta^{***} \leq \delta^{**} \leq \delta^{*} \leq \frac{\pi}{2}$ (see Fig. 10*a*), let $\delta^{*} \to \delta^{**}$ in Eq. 41. For the case of $0 \leq \delta^{*} \leq \delta^{***} \leq \delta^{**} \leq \frac{\pi}{2}$ (see Fig. 10*c*), let $\delta^{*} \to \delta^{***}$ in this equation.

Equation 41 contains four basic integrals which can be defined as

$$\left.\begin{aligned}
[I_1(\alpha,\phi)]_{a}^{b} &= \int_{a}^{b} \frac{\sin\phi\sin\delta + \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta]^3}\,d\delta \\
[I_2(\alpha,\phi)]_{a}^{b} &= \int_{a}^{b} \frac{\sin\phi\sin\delta + \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta][1-\sin^2\phi\sin^2\delta]}\,d\delta \\
[I_3(\alpha,\phi)]_{a}^{b} &= \int_{a}^{b} \frac{\sin\phi\sin\delta - \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta]^3}\,d\delta \\
[I_4(\alpha,\phi)]_{a}^{b} &= \int_{a}^{b} \frac{\sin\phi\sin\delta - \cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta][1-\sin^2\phi\sin^2\delta]}\,d\delta
\end{aligned}\right\} \quad (42)$$

Because of the combinations in which these integrals occur, the work can be simplified by further defining

$$\left.\begin{aligned}
[I_5(\alpha,\phi)]_{a}^{b} &= [I_1(\alpha,\phi)]_{a}^{\delta^{*}} + [I_3(\alpha,\phi)]_{\delta^{*}}^{b} \\
[I_6(\alpha,\phi)]_{a}^{b} &= [I_2(\alpha,\phi)]_{a}^{\delta^{*}} + [I_4(\alpha,\phi)]_{\delta^{*}}^{b}
\end{aligned}\right\}. \quad (43)$$

Using Eqs. 42 and 43, Eq. 41 becomes

$$\frac{Mom_{\text{petals}}}{\rho h_c a_1^2 v \tan\alpha} = \left(\frac{x}{l_1}\right)^2 [I_5(\alpha,\phi)]_{\delta^{***}}^{\delta^{**}} - [I_6(\alpha,\phi)]_{\delta^{***}}^{\delta^{**}}, \quad (44)$$

which holds for any combination of δ^{*}, δ^{**}, δ^{***}.

VII. PETAL SEPARATION

In this section a criterion for evaluation of δ^{***} will be established. Then the exact expression for δ^{***} and the associated petal momentum will be derived.

a. Theorem: Each petal element, defined by its position δ, starts to separate from the extended cone generator at the instant the base circle intersects the target plane at the position δ associated with that particular element. The base circle is defined by the intersection of the cylindrical and conical parts of the projectile.

Proof: The discussion associated with assumption (1) leads to the separation criterion

$$\frac{d}{dt}\,|Mom_{\text{petals}}| \leq 0 \quad (45)$$

Since ξ is independent of x for a moving particle, all petal elements travel with constant momentum. Thus the petal element momentum increases only because of increasing petal element mass. With reference

to Fig. 11, since the mass of the forward petal increases until the base circle and target intersect, no separation can occur before that event. Equation 45 is the precise definition of separation, so that even though the rearward petal has a region out of contact with the cone (in its dotted position), it is not classified as separated. In fact, since there

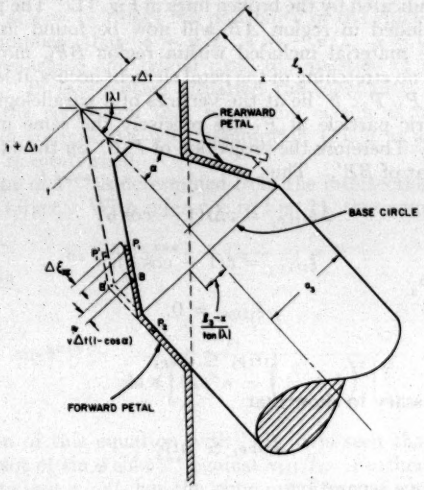


FIG. 11.

are no retarding forces on the particles in this non-touching region, these particles continue with constant velocity and remain on the cone generator extension. Thus the rearward petal cannot separate before the base circle intersects the target plane.

It is now only necessary to show that the petals are separated immediately after this intersection. If this is so, it can be concluded that the separation occurs at the time of intersection. It is obvious that no further consideration need be given to the rearward petals because after the base circle passes through the target $\frac{d}{dt}|Mom_{petals}| = 0$ and remains so. Also, it is apparent from Fig. 11 that there is no physical contact with the projectile.

Because of the constant body radius of the projectile the forward petal element mass must remain constant after intersection if no separation occurs. Hence separation will be proven by first assuming no separation and then showing that particle by particle the velocities will

be lower after intersection. This will lead to a decrease in momentum and a contradiction of the assumption of no separation.

With reference to Fig. 11, the velocity for particle P_1 is obtained in the following manner:

In an interval of time Δt the projectile will advance a distance $v\Delta t$ to the position indicated by the broken lines in Fig. 11. The petal material originally included in region AB will now be found in region AB'. Similarly, the material included within region BP_1, moves to region $B'P_1'$. Since no stretching of the petal element occurs, it is evident that the points B, P_1, P_1', B' lie at the vertices of a parallelogram. Therefore the generic particle at P_1 has precisely the same motion as the particle at B. Therefore the projection of P_1P_1' on the axis of the cone is equal to that of BB'. Thus

$$\Delta\xi_{III_{P_1}} = v\Delta t(1 - \cos\alpha)$$

and

$$\dot{\xi}_{III_{P_1}} = v(1 - \cos\alpha). \tag{46}$$

For particle P_2

$$\dot{\xi}_{III_{P_2}} = 0. \tag{47}$$

Thus since

$$\dot{\xi}_{III_{P_2}} \leq \dot{\xi}_{II_{P_2}} \tag{48}$$

it is only necessary to show that

$$\dot{\xi}_{III_{P_1}} \leq \dot{\xi}_{II_{P_1}} \tag{49}$$

in order to prove separation.

If the equality sign holds in both Eqs. 48 and 49, the question of separation becomes indeterminate. By Eqs. 37, 46 and 49 the condition to be proven becomes

$$(1 - \cos\alpha) \leq \frac{\tan\alpha(\sin\phi\sin\delta + \cos\alpha)}{(1 - \sin^2\phi\sin^2\delta)^{1/2} + \tan\alpha\sin\phi\sin\delta}, \tag{50}$$

where

$$0 < \alpha \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad -\frac{\pi}{2} \leq \delta \leq \frac{\pi}{2}.$$

A new variable η is defined as

$$\eta = \sin\phi\sin\delta, \quad -1 \leq \eta \leq 1 \tag{51}$$

Equation 50 becomes

$$(1 - \cos\alpha) \leq \frac{\tan\alpha(\eta + \cos\alpha)}{(1 - \eta^2)^{1/2} + \eta\tan\alpha}. \tag{52}$$

Examination of Eq. 52 reveals that no relative maximum exists. Using the values at the end points of $-1 \leq \eta \leq 1$, the inequality becomes

$$1 - \cos\alpha \leq 1 + \frac{1}{\eta}\cos\alpha \tag{53}$$

and thus only at $\eta = -1$ does the possibility of the equality exist.

$\eta = -1$ implies that $\phi = \pi/2$, $\delta = -\pi/2$, $\alpha = 0$. For this physically unrealizable and trivial case, the question of separation is indeterminate. In every other situation the inequality (53) is valid and leads to the conclusion that separation occurs as hypothesized. As discussed in conjunction with assumption (1), it is assumed that contact is not re-established.

b. The value of δ^{***} is determined from the intersection of the base circle and the target. With reference to Fig. 11, this requires that

$$a_3 \tan|\lambda^{***}| - (l_3 - x) = 0$$

or using Eq. 28,

$$\sin\delta^{***} = \frac{\dfrac{x}{l_3} - 1}{\sin\phi\left[\tan^2\alpha + \left(\dfrac{x}{l_3} - 1\right)^2\right]^{1/2}}. \tag{54}$$

By comparison of this equation with (15), it is seen that Fig. 6 also represents a plot of $\sin\phi\sin\delta^{***}$ against x_{III}/l_3. Furthermore, Eqs. 8 and 16 indicate that x_{III}/l_3 has the same range as x_1/l_1.

c. The value of the petal momentum at separation is the total momentum transferred to the non-separated petal. Thus since $\xi_{III} = \xi_{II} = \xi$, the momentum in the separated petal is

$$Mom_{sep} = 2\left[\int_{-\pi/2}^{\delta^*} \xi_f dM_{III} + \int_{\delta^*}^{\delta^{***}} \xi_r dM_{III}\right], \tag{55}$$

where ξ_f and ξ_r are given by Eqs. 37 and 39, respectively, and dM_{III} is obtained from Eq. 34 as

$$dM_{III} = \rho h_o\left(\frac{a''^2 - a'^2}{2}\right)d\delta. \tag{56}$$

When $\delta^{***} < \delta^*$, let $\delta^* = \delta^{***}$ in Eq. 55.

By reasoning similar to that leading to Eq. 14

$$a''(\delta) = \frac{a_3}{(1 - \sin^2\delta\sin^2\phi)^{1/2}}. \tag{57}$$

Thus Eq. 15 can be written as:

$$\frac{Mom_{sep}}{\rho h_e(a_3^2-a_1^2)v\tan\alpha}=\int_{-\pi/2}^{\delta^*}\frac{\sin\phi\sin\delta+\cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2}+\tan\alpha\sin\phi\sin\delta][1-\sin^2\phi\sin^2\delta]}d\delta$$
$$+\int_{\delta^*}^{\delta^{***}}\frac{\sin\phi\sin\delta-\cos\alpha}{[(1-\sin^2\phi\sin^2\delta)^{1/2}+\tan\alpha\sin\phi\sin\delta][1-\sin^2\phi\sin^2\delta]}d\delta. \quad (58)$$

By the definitions for I_2, I_4 and I_6 of Eqs. 42 and 43, this becomes

$$\frac{Mom_{sep}}{\rho h_e(a_3^2-a_1^2)v\tan\alpha}=[I_6(\alpha,\phi)]_{-\pi/2}^{\delta^{***}}. \quad (59)$$

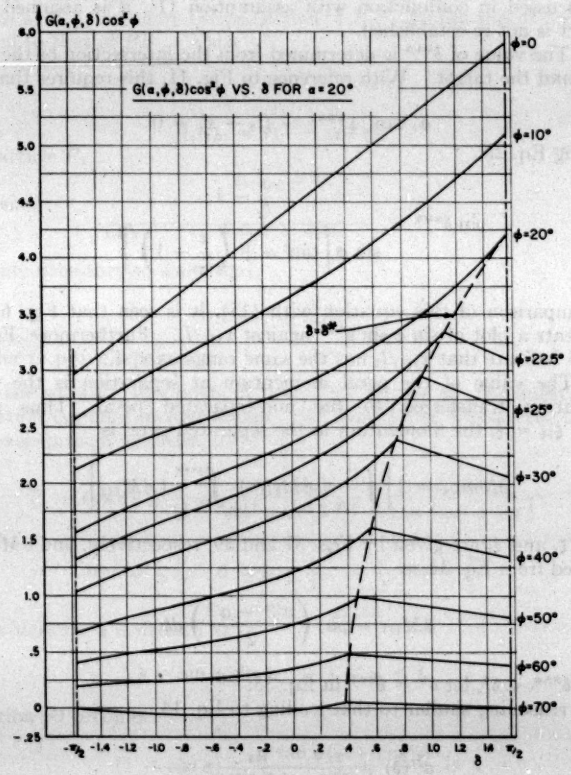


FIG. 12.

VIII. PROJECTILE VELOCITY–DISTANCE RELATIONSHIP

By consideration of conservation of momentum within the zone of action,

$$\Delta v = v_s - v = \frac{Mom_{\text{total}}}{m} \tag{60}$$

where $Mom_{\text{total}} = Mom_{\text{slug}} + Mom_{\text{petals}} + Mom_{\text{sep}}$.

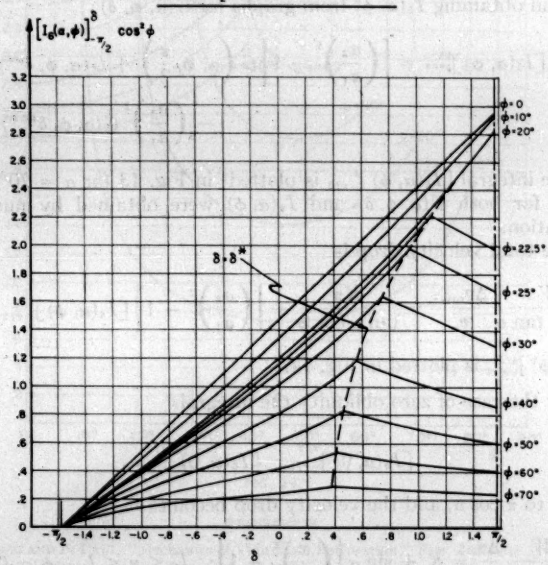


FIG. 13.

Using Eqs. 23, 44 and 59 in Eq. 60, and considering $\Delta v \ll v_s$, the following equation for velocity change is obtained:

$$\frac{W}{\gamma h_s a_1^2 \tan\alpha}\frac{\Delta v}{v_s} = \frac{KA(\delta^{**})}{a_1^2 \tan\alpha} + \left(\frac{x}{l_1}\right)^2 \left[I_5(\alpha,\phi)\right]_{\delta^{**}}^{\delta^{***}} - \left[I_6(\alpha,\phi)\right]_{-\pi/2}^{\delta^{**}} + \left(\frac{a_3}{a_1}\right)^2 \left[I_6(\alpha,\phi)\right]_{-\pi/2}^{\delta^{***}}. \tag{61}$$

The term $\left(\frac{x}{l_1}\right)^2 \left[I_5(\alpha,\phi)\right]_{\delta^{**}}^{\delta^{***}}$ can be obtained from the function

$G(\alpha, \phi, \delta^{**})$, which is plotted[6] in Fig. 12 for $\alpha = 20°$. $G(\alpha, \phi, \delta^{**})$ is defined as

$$G(\alpha, \phi, \delta^{**}) = 2(1 - \tan\alpha \tan\phi)^2 [I_5(\alpha, \phi)]_{-\pi/2}^{\pi/2} - \left(\frac{x}{l_1}\right)^2 [I_5(\alpha, \phi)]_{-\pi/2}^{\pi/2}. \quad (62)$$

By comparison of Eqs. 15 and 54, it is seen that these graphs also represent the function $G(\alpha, \phi, \delta^{***})$. The following relationship is useful in obtaining $I_5(\alpha, \phi)$ from graphs for $G(\alpha, \phi, \delta)$:

$$\left(\frac{x}{l_1}\right)^2 [I_5(\alpha, \phi)]_{\delta^{***}}^{\delta^{***}} = \left[\left(\frac{a_2}{a_1}\right)^2 - 1\right] G\left(\alpha, \phi, \frac{\pi}{2}\right) + G(\alpha, \phi, \delta^{**}) - \left(\frac{a_2}{a_1}\right)^2 G(\alpha, \phi, \delta^{***}). \quad (63)$$

The integral $[I_5(\alpha, \phi)]_{-\pi/2}^{\delta}$ is plotted[7] in Fig. 13 for $\alpha = 20°$. The values for both $G(\alpha, \phi, \delta)$ and $I_5(\alpha, \phi)$ were obtained by numerical integration.

The total velocity drop is

$$\frac{W}{\gamma h_a a_1^2 \tan\alpha} \frac{\Delta v_{\text{total}}}{v_s} = \frac{K\pi}{\tan\alpha \cos\phi} + \left[\left(\frac{a_2}{a_1}\right)^2 - 1\right] [I_5(\alpha, \phi)]_{-\pi/2}^{\pi/2}. \quad (64)$$

$[I_5(\alpha, \phi)]_{-\pi/2}^{\pi/2}$ is plotted in Fig. 14.

For the case of zero obliquity the integrals

$$[I_5(\alpha, 0)]_{-\pi/2}^{\pi/2}, \quad [I_5(\alpha, 0)]_{-\pi/2}^{\pi/2}$$

reduce to $\pi \cos\alpha$, and the velocity drop becomes:

$$\frac{W}{\pi\gamma h_a a_1^2} \frac{\Delta v}{v_s} = K + \sin\alpha \left[\left(\frac{x}{l_1}\right)^2 - 1\right]; \quad l_1 \le x \le l_2, \quad \phi = 0 \quad (65)$$

which leads to

$$\frac{W}{\pi\gamma h_a a_1^2} \frac{\Delta v_{\text{total}}}{v_s} = K + \sin\alpha \left[\left(\frac{a_2}{a_1}\right)^2 - 1\right], \quad \phi = 0. \quad (66)$$

[6] See HEAP Progress Report No. 8, Bulova Research and Development Laboratories, 15 July, 1956, U. S. Army Contract DAI-36-038-507-ORD(P)-339 for complete graphs of $G(\alpha, \phi, \delta^{**})$.

[7] See HEAP Progress Report No. 8, Bulova Research and Development Laboratories, 15 July, 1956, U. S. Army Contract DAI-36-038-507-ORD(P)-339 for complete graphs of $I_5(\alpha, \phi)$.

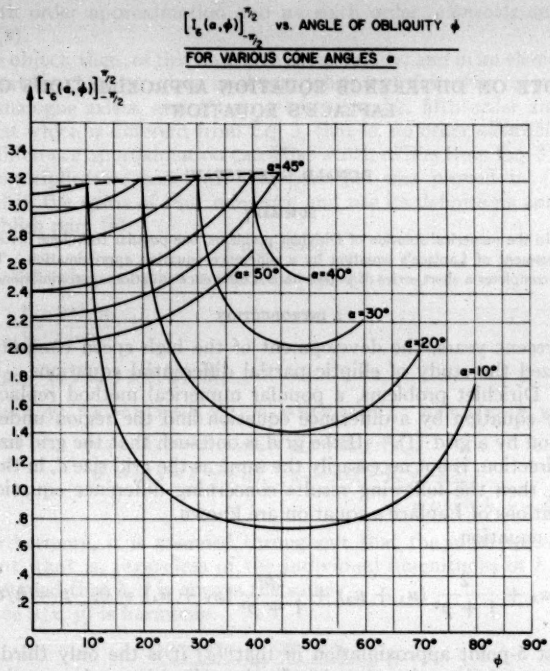


FIG. 14.

68
9

REFERENCES

(1) M. ZAID AND B. PAUL, "Mechanics of Thin Plate Perforation," *Proc. 2nd Fuze Symposium*, Diamond Ordnance Fuze Labs., March, 1956.

(2) M. ZAID AND B. PAUL, "Armor Penetration," *Ordnance*, Vol. XLI, pp. 609–611 (1957).

(3) M. ZAID AND B. PAUL, "Mechanics of High-Speed Projectile Perforation," JOUR. FRANKLIN INST., Vol. 264, pp. 117–126 (1957).

(4) B. PAUL AND M. ZAID, "Normal Perforation of a Thin Plate by Truncated Projectiles," JOUR. FRANKLIN INST., Vol. 265, pp. 317–335 (1958).

(5) JOSEPH M. KRAFT, "Surface Friction in Ballistic Penetration," *J. Appl. Phys.*, Vol. 26, No. 10 (1955).

NOTE ON DIFFERENCE EQUATION APPROXIMATIONS OF LAPLACE'S EQUATION*

BY

DONALD GREENSPAN[1]

SUMMARY

In the numerical solution of Dirichlet problems, one popular technique involves replacement of Laplace's equation by a difference equation approximation. This note completes a short series of papers on *best* difference equation approximations.

1. INTRODUCTION

In recent years, the development of the high speed computer has revitalized the study of elliptic partial differential equations. In the case of Dirichlet problems, a popular numerical method replaces the Laplace equation by a difference equation and the region under consideration by a grid (1).[2] If the grid is one such that the grid size h, in the x-direction, is not necessarily the same as the grid size d, in the y-direction, then the following results concerning difference equation approximations of Laplace's equation are known.

The equation

$$-4u_0+\frac{2}{1+p^2}(u_1+u_3)+\frac{2p^2}{1+p^2}(u_2+u_4)=0, \quad p=h/d \tag{1}$$

is a best 5-point approximation in that (a) it is the only third order analogue, and (b) no fourth order, 5-point, *allowable* analogue exists (2). This result is independent of whether h and d are equal or not.

If $h=d$, the equation

$$-20u_0+4(u_1+u_2+u_3+u_4)+u_5+u_6+u_7+u_8=0 \tag{2}$$

is a best nine-point approximation in that (a) it is the only seventh order analogue, and (b) no eighth order, nine-point, *allowable* analogue exists (3, 4).

If $h\neq d$, then the equation

$$-20u_0+2\frac{5-p^2}{1+p^2}(u_1+u_3)+2\frac{5p^2-1}{1+p^2}(u_2+u_4)+(u_5+u_6+u_7+u_8)=0 \tag{3}$$

* This work was done while the author was a summer faculty associate of the Transport Division, Mathematical Services Unit, Boeing Airplane Company, Renton, Washington.

[1] Department of Mathematics and Statistics, Purdue University, Lafayette, Ind.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

is a fifth order approximation and no sixth order, *allowable* analogue exists (5).

The object, then, of this note is to show, finally, and in an elementary way, that Eq. 3 is also a best analogue, when $h \neq d$, in that (*a*) no sixth order analogue exists, and (*b*) no other *allowable*, fifth order analogue can exist which is different from Eq. 3, that is, no other *allowable*, fifth order difference approximation can exist which differs from Eq. 3 except by a multiplicative constant. As part (*a*) was proved in (5), we proceed in the spirit of that reference and use its definitions and ideas to establish part (*b*).

2. DISCUSSION

The following notation is used throughout:

$$A_{m,n} = \frac{1}{m!n!}\frac{\partial^{m+n}u}{\partial x^m \partial y^n}, \tag{4}$$

$$h = pd \quad (h>0,\ p>0,\ d>0), \tag{5}$$

$$u_0 = u(x,y), \quad u_1 = u(x+h,y), \quad u_2 = u(x,y+d), \quad u_3 = u(x-h,y),$$
$$u_4 = (x,y-d), \quad u_5 = u(x+h,y+d),$$
$$u_6 = u(x-h,y+d), \quad u_7 = u(x-h,y-d),$$
$$u_8 = u(x+h,y-d). \tag{6}$$

Furthermore, it is assumed throughout that the mesh ratio p is a constant, that is, regardless of the individual magnitudes of h and d, it is assumed that h/d is always the same.

Since $u(x, y)$ is harmonic

$$\frac{\partial^{m+n+2}u}{\partial x^{m+2}\partial y^n} + \frac{\partial^{m+n+2}u}{\partial x^m \partial y^{n+2}} = 0 \tag{7}$$

for m, n non-negative integers. In terms of the $A_{m,n}$, the relationships of Eq. 7 may be enumerated, after some manipulation (3), as follows:

$$\begin{aligned}
A_{2,0} &= -A_{0,2} & A_{3,2} &= -2A_{1,4} \\
A_{3,0} &= -\frac{1}{3}A_{1,2} & A_{2,3} &= -10A_{0,5} \\
A_{2,1} &= -3A_{0,3} & A_{6,0} &= -A_{0,6} \\
A_{4,0} &= A_{0,4} & A_{5,1} &= A_{1,5} \\
A_{3,1} &= -A_{1,3} & A_{4,2} &= 15A_{0,6} \\
A_{2,2} &= -6A_{0,4} & A_{3,3} &= -\frac{10}{3}A_{1,5} \\
A_{5,0} &= \frac{1}{5}A_{1,4} & A_{2,4} &= -15A_{0,6} \\
A_{4,1} &= 5A_{0,5}
\end{aligned} \tag{8}$$

Now let

$$L(u) = \sum_{0}^{8} \alpha_i u_i. \tag{9}$$

It was shown in (5) by use of Taylor series and Eqs. 4 and 5 that Eq. 9 can be rewritten as follows:

$$\begin{aligned}
L(u) = {} & A_{0,0}(\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8) \\
& +d\{A_{1,0}p(\alpha_1-\alpha_3+\alpha_5-\alpha_6-\alpha_7+\alpha_8) \\
& +A_{0,1}(\alpha_2-\alpha_4+\alpha_5+\alpha_6-\alpha_7-\alpha_8)\} \\
& +d^2\{A_{0,2}[-\alpha_1p^2+\alpha_2-\alpha_3p^2+\alpha_4+(1-p^2)(\alpha_5+\alpha_6+\alpha_7+\alpha_8)] \\
& +A_{1,1}p(\alpha_5-\alpha_6+\alpha_7-\alpha_8)\} \\
& +d^3\left\{\frac{p}{3}A_{1,2}[-\alpha_1p^2+\alpha_3p^2+(3-p^2)(\alpha_5-\alpha_6-\alpha_7+\alpha_8)]\right. \\
& \left.+A_{0,3}[\alpha_2-\alpha_4+(1-3p^2)(\alpha_5+\alpha_6-\alpha_7-\alpha_8)]\right\} \\
& +d^4\{A_{1,3}(p-p^3)(\alpha_5-\alpha_6+\alpha_7-\alpha_8) \\
& +A_{0,4}[\alpha_1p^4+\alpha_2+\alpha_3p^4+\alpha_4+(p^4-6p^2+1)(\alpha_5+\alpha_6+\alpha_7+\alpha_8]\} \\
& +d^5\left\{\frac{p}{5}A_{1,4}[\alpha_1p^4-\alpha_3p^4+(p^4-10p^2+5)(\alpha_5-\alpha_6-\alpha_7+\alpha_8)\right. \\
& \left.+A_{0,5}[\alpha_2-\alpha_4+(5p^4-10p^2+1)(\alpha_5+\alpha_6-\alpha_7-\alpha_8)]\right\}+0(d^6).
\end{aligned} \tag{10}$$

Define now the set A^* to be the set whose elements are $A_{0,0}$, $A_{1,0}$, $A_{0,1}$, $A_{1,1}$, $A_{0,2}$, $A_{1,2}$, $A_{0,3}$, $A_{1,3}$, $A_{0,4}$, $A_{1,4}$, $A_{0,5}$ and consider the following table which relates various harmonic functions to the elements of A^*.

Harmonic Function	Values of Elements of A^* at (0, 0)
1. $u = 1$	All zero except $A_{0,0}$
2. $u = y$	All zero except $A_{0,1}$
3. $u = x$	All zero except $A_{1,0}$
4. $u = xy$	All zero except $A_{1,1}$
5. $u = x^2 - y^2$	All zero except $A_{0,2}$
6. $u = \frac{xy^2}{2} - \frac{x^3}{3}$	All zero except $A_{1,2}$
7. $u = \frac{y^3}{-6} + \frac{yx^2}{2}$	All zero except $A_{0,3}$

8. $u = \frac{xy^3}{6} - \frac{x^3y}{6}$ — All zero except $A_{1,3}$

9. $u = \frac{y^4}{24} - \frac{x^2y^2}{4} + \frac{x^4}{24}$ — All zero except $A_{0,4}$

10. $u = \frac{xy^4}{24} - \frac{x^3y^2}{12} + \frac{x^5}{120}$ — All zero except $A_{1,4}$

11. $u = \frac{y^5}{120} - \frac{y^3x^2}{12} + \frac{yx^4}{24}$ — All zero except $A_{0,5}$

If one lets $(x, y) = (0, 0)$ and considers the first harmonic function above, then Eq. 10 reduces to

$$L(u) = A_{0,0}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8).$$

If one wishes to make $L(u)$ of order five in d, it must follow then, since $A_{0,0}$ is constant, that

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = \epsilon_1 = 0(d^6).$$

Using this idea on each of the harmonic functions listed above, it follows that if Eq. 10 is to be of order five in d, that each of the following equations must be satisfied:

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8=\epsilon_1=0(d^6) \tag{11}$$

$$\alpha_1-\alpha_3+\alpha_5-\alpha_6-\alpha_7+\alpha_8=\epsilon_2=0(d^5) \tag{12}$$

$$\alpha_2-\alpha_4+\alpha_5+\alpha_6-\alpha_7-\alpha_8=\epsilon_3=0(d^5) \tag{13}$$

$$-\alpha_1p^2+\alpha_2-\alpha_3p^2+\alpha_4+(1-p^2)(\alpha_5+\alpha_6+\alpha_7+\alpha_8)=\epsilon_4=0(d^4) \tag{14}$$

$$\alpha_5-\alpha_6+\alpha_7-\alpha_8=\epsilon_5=0(d^4) \tag{15}$$

$$-\alpha_1p^2+\alpha_3p^2+(3-p^2)(\alpha_5-\alpha_6-\alpha_7+\alpha_8)=\epsilon_6=0(d^3) \tag{16}$$

$$\alpha_2-\alpha_4+(1-3p^2)(\alpha_5+\alpha_6-\alpha_7-\alpha_8)=\epsilon_7=0(d^3) \tag{17}$$

$$\alpha_1p^4+\alpha_2+\alpha_3p^4+\alpha_4+(p^4-6p^2+1)(\alpha_5+\alpha_6+\alpha_7+\alpha_8)=\epsilon_8=0(d^2) \tag{18}$$

$$\alpha_1p^4-\alpha_3p^4+(p^4-10p^2+5)(\alpha_5-\alpha_6-\alpha_7+\alpha_8)=\epsilon_9=0(d) \tag{19}$$

$$\alpha_2-\alpha_4+(5p^4-10p^2+1)(\alpha_5+\alpha_6-\alpha_7-\alpha_8)=\epsilon_{10}=0(d). \tag{20}$$

However, since

$$\frac{\epsilon_2}{3}(5p^2 - 7p^4) + \frac{\epsilon_6}{3}(5 - 10p^2) = \epsilon_9 \tag{21}$$

and

$$\frac{\epsilon_7}{3}(10 - 5p^2) + \frac{\epsilon_3}{3}(5p^2 - 7) = \epsilon_{10} \tag{22}$$

it follows that one need only consider Eqs. 11–18, subject to these further restrictions in the ϵ_i's. The solution of the system (11–18), in terms of α_0, is given by:

$$\begin{aligned}
\alpha_1 &= [30p^2(p^2+1)]^{-1}[3p^2(5-p^2)(\epsilon_1-\alpha_0) \\
&\quad + 5p^2(p^2+1)(3-p^2)\epsilon_2 + 3p^2(p^2-6)\epsilon_4 - 5p^2(p^2+1)\epsilon_6 + 3p^2\epsilon_8] \\
\alpha_2 &= [30p^2(p^2+1)]^{-1}[3p^2(5p^2-1)(\epsilon_1-\alpha_0) \\
&\quad - 5(p^2+1)(1-3p^2)\epsilon_3 + 3(6p^2-1)\epsilon_4 + 5(p^2+1)\epsilon_7 + 3\epsilon_8] \\
\alpha_3 &= [30p^2(p^2+1)]^{-1}[3p^2(5-p^2)(\epsilon_1-\alpha_0) \\
&\quad - 5p^2(p^2+1)(3-p^2)\epsilon_2 + 3p^2(p^2-6)\epsilon_4 + 5p^2(p^2+1)\epsilon_6 + 3p^2\epsilon_8] \\
\alpha_4 &= [30p^2(p^2+1)]^{-1}[3p^2(5p^2-1)(\epsilon_1-\alpha_0) \\
&\quad + 5(p^2+1)(1-3p^2)\epsilon_3 + 3(6p^2-1)\epsilon_4 - 5(p^2+1)\epsilon_7 + 3\epsilon_8] \\
\alpha_5 &= \tfrac{1}{2}[30p^2(p^2+1)]^{-1}[3p^2(p^2+1)(\epsilon_1-\alpha_0) \\
&\quad + 5p^4(p^2+1)\epsilon_2 + 5(p^2+1)\epsilon_3 - 3(p^2+1)(p^2-1)\epsilon_4 \\
&\quad + 15p^2(p^2+1)\epsilon_5 + 5p^2(p^2+1)\epsilon_6 - 5(p^2+1)\epsilon_7 - 3(p^2+1)\epsilon_8] \\
\alpha_6 &= \tfrac{1}{2}[30p^2(p^2+1)]^{-1}[3p^2(p^2+1)(\epsilon_1-\alpha_0) \\
&\quad - 5p^4(p^2+1)\epsilon_2 + 5(p^2+1)\epsilon_3 - 3(p^2+1)(p^2-1)\epsilon_4 \\
&\quad - 15p^2(p^2+1)\epsilon_5 - 5p^2(p^2+1)\epsilon_6 - 5(p^2+1)\epsilon_7 - 3(p^2+1)\epsilon_8] \\
\alpha_7 &= \tfrac{1}{2}[30p^2(p^2+1)]^{-1}[3p^2(p^2+1)(\epsilon_1-\alpha_0) \\
&\quad - 5p^4(p^2+1)\epsilon_2 - 5(p^2+1)\epsilon_3 - 3(p^2+1)(p^2-1)\epsilon_4 \\
&\quad + 15p^2(p^2+1)\epsilon_5 - 5p^2(p^2+1)\epsilon_6 + 5(p^2+1)\epsilon_7 - 3(p^2+1)\epsilon_8] \\
\alpha_8 &= \tfrac{1}{2}[30p^2(p^2+1)]^{-1}[3p^2(p^2+1)(\epsilon_1-\alpha_0) \\
&\quad + 5p^4(p^2+1)\epsilon_2 - 5(p^2+1)\epsilon_3 - 3(p^2+1)(p^2-1)\epsilon_4 \\
&\quad - 15p^2(p^2+1)\epsilon_5 + 5p^2(p^2+1)\epsilon_6 + 5(p^2+1)\epsilon_7 - 3(p^2+1)\epsilon_8].
\end{aligned}$$

Substitution of these latter values into Eq. 9 and recombination yields:

$$
\begin{aligned}
L(u) = [30p^2(1+p^2)]^{-1}\Big\{&\alpha_0\Big[30u_0p^2(1+p^2) - 3p^2(5-p^2)(u_1+u_3) \\
&- 3p^2(5p^2-1)(u_2+u_4) - \frac{3p^2(1+p^2)}{2}(u_5+u_6+u_7+u_8)\Big] \\
&+\epsilon_1\Big[3p^2(5-p^2)(u_1+u_3) + 3p^2(5p^2-1)(u_2+u_4) \\
&\qquad + \frac{3p^2(p^2+1)}{2}(u_5+u_6+u_7+u_8)\Big] \\
&+\epsilon_2\Big[5p^2(p^2+1)(3-p^2)(u_1-u_3) + \frac{5p^4(p^2+1)}{2}(u_5-u_6-u_7+u_8)\Big] \\
&+\epsilon_3\Big[5(p^2+1)(1-3p^2)(u_4-u_2) + \frac{5(p^2+1)}{2}(u_5+u_6-u_7-u_8)\Big] \\
&+\epsilon_4\Big[3p^2(p^2-6)(u_1+u_3) + 3(6p^2-1)(u_2+u_4) \\
&\qquad - \frac{3(p^2+1)(p^2-1)}{2}(u_5+u_6+u_7+u_8)\Big] \\
&+\epsilon_5\Big[\frac{15}{2}p^2(p^2+1)(u_5-u_6+u_7-u_8)\Big] \\
&+\epsilon_6\Big[-5p^2(p^2+1)(u_1-u_3) + \frac{5p^2(p^2+1)}{2}(u_5-u_6-u_7+u_8)\Big] \\
&+\epsilon_7\Big[5(p^2+1)(u_2-u_4) - \frac{5(p^2+1)}{2}(u_5+u_6-u_7-u_8)\Big] \\
&+\epsilon_8\Big[3p^2(u_1+u_3) + 3(u_2+u_4) - \frac{3(p^2+1)}{2}(u_5+u_6+u_7+u_8)\Big]\Big\}.
\end{aligned}
\tag{23}
$$

However, from the Taylor series, in p and d only, for u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 (5), Eq. 23 may be rewritten as

$$
\begin{aligned}
L(u) = [30p^2(p^2+1)]^{-1}\Big\{&\alpha_0\Big[30u_0p^2(p^2+1) \\
&- 3p^2(5-p^2)(u_1+u_3) - 3p^2(5p^2-1)(u_2+u_4) \\
&- \frac{3p^2(p^2+1)}{2}(u_5+u_6+u_7+u_8)\Big] + \epsilon_1\cdot 0(d^0) \\
&+ \epsilon_2\cdot 0(d) + \epsilon_3\cdot 0(d) + \epsilon_4\cdot 0(d^2) + \epsilon_5\cdot 0(d^2) + \epsilon_6\cdot 0(d^3) \\
&+ \epsilon_7\cdot 0(d^3) + \epsilon_8\cdot 0(d^4)\Big\}.
\end{aligned}
\tag{24}
$$

From the definition of the ϵ_i's, however, Eq. 24 may be rewritten as

$$L(u) = [30p^2(p^2+1)]^{-1}\left\{\alpha_0\left[30u_0p^2(p^2+1) - 3p^2(5-p^2)(u_1+u_3) - 3p^2(5p^2-1)(u_2+u_4) - \frac{3p^2(p^2+1)}{2}(u_5+u_6+u_7+u_8)\right] + 0(d^6)\right\}. \quad (25)$$

However, from Eq. 10, it follows that

$$L(u) = 0(d^6). \quad (26)$$

Finally, from Eqs. 25 and 26 it follows that

$$[30p^2(p^2+1)]^{-1}\left\{\alpha_0\left[30u_0p^2(p^2+1) - 3p^2(5-p^2)(u_1+u_3) - 3p^2(5p^2-1)(u_2+u_4) - \frac{3p^2(p^2+1)}{2}(u_5+u_6+u_7+u_8)\right]\right\} \equiv 0(d^6). \quad (27)$$

Hence, if one is to have a difference approximation which is of fifth order, it follows from Eq. 27 that it must be of the form

$$[30p^2(p^2+1)]^{-1}\left\{\alpha_0\left[30u_0p^2(p^2+1) - 3p^2(5-p^2)(u_1+u_3) - 3p^2(5p^2-1)(u_2+u_4) - \frac{3p^2(p^2+1)}{2}(u_5+u_6+u_7+u_8)\right]\right\} = 0,$$

which is equivalent, by means of a multiplicative constant, to Eq. 3, as was to be proved.

REFERENCES

(1) D. R. HARTREE, "Numerical Analysis," Oxford, 1952, pp. 215–219, 221–232.
(2) D. GREENSPAN, "On a 'Best' Five-Point Difference Analogue of Laplace's Equation," JOUR. FRANKLIN INST., Vol. 266, pp. 39–45 (1958).
(3) D. GREENSPAN, "On a 'Best' 9-Point Difference Equation Analogue of Laplace's Equation," JOUR. FRANKLIN INST., Vol. 263, pp. 425–430 (1957).
(4) D. GREENSPAN, "Note on Nine-Point Analogues of Laplace's Equation," JOUR. FRANKLIN INST., Vol. 264, pp. 453–455 (1957).
(5) D. GREENSPAN, "On 'Best' Nine-Point Laplace Difference Analogues on Rectangular Grids," *QJMAM*, Vol. XII, Pt. 1, pp. 111–116 (1959).

CREEP STRESSES AND STRAINS IN AN AXIALLY LOADED PLATE WITH A HOLE*

BY

JOSEPH MARIN[1]

SUMMARY

This paper develops an approximate method for determining the stresses and strains in a plate with a circular hole subjected to axial tension and accompanied by creep (Fig. 1). Plates with holes subjected to creep are found in bolted and riveted connections of aircraft and other constructions operating at elevated temperatures. Lack of symmetry makes an exact analysis of this problem difficult. For this reason an approximate theory was developed.

INTRODUCTION

In considering possible methods of determining the creep stress distribution in an axially loaded plate with a circular hole, methods used for plastic stress distribution as given in (1)[2] and (3) were reviewed. The complexities of the "exact" methods used for the plastic range and the additional difficulties encountered in the case of creep discouraged the writer from using an "exact" method. For these reasons, the following approximate method of solution was developed.

STRESSES IN A PLATE WITH A HOLE AXIALLY LOADED AND ACCOMPANIED BY CREEP

An approximate method of determining the creep-stress distribution across the net sections AB and CD of the plate in Fig. 1 will first be developed. The stress on sections AB and CD can be found by considering forces acting on the ring $AEDF$. Since this ring is symmetrically loaded, only one quadrant of the ring, as shown in Fig. 2, need be considered. The applied forces F in Fig. 1 will produce axial forces of varying magnitude over the circular boundary AE of Fig. 2. In addition to these forces, reactions M_0 and N_0 are applied over the cross-section, AB. The ring sector in Fig. 2 may now be considered as a curved bar fixed at cross-section EG and subjected to vertical forces over the arc AE and to forces M_0 and N_0 at cross-section AB. The force N_0 can be determined by statics. However, the moment M_0 is statically indeterminate and its value can be obtained from the condition that there is no relative rotation between cross-sections AB and EG.

* The study reported in this paper was made possible through the sponsorship and financial assistance of Wright Air Development Center.

[1] Department of Engineering Mechanics, The Pennsylvania State University, University Park, Pa.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

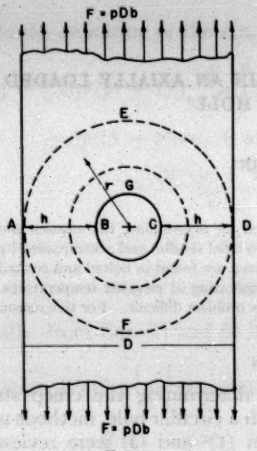


FIG. 1.

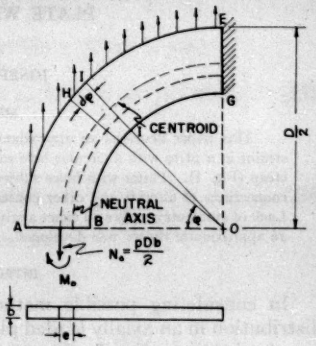


FIG. 2.

Determination of M_0

Figure 3 shows a differential length of the curved bar in Fig. 2 corresponding to an angle $d\phi$. The problem, therefore, reduces to finding the creep stresses in a curved bar subjected to bending and axial loading. To determine these stresses, it is first necessary to realize that the neutral and centroidal axis of the beam cross-section do not coincide. If the distance between these axes is γ, then for plane cross-sections to remain plane

$$\overline{PQ} = (y - \gamma)\Delta d\phi = \epsilon_z(r - y)d\phi$$

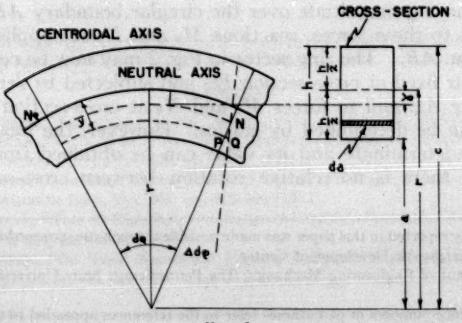


FIG. 3.

or

$$\epsilon_x = \left(\frac{y - \gamma}{r - y}\right)\frac{\Delta d\phi}{d\phi} \qquad (a)$$

where

ϵ_x = the unit elongation for the fiber at distance y
$\Delta d\phi$ = relative rotation of the beam cross-sections for the differential length of beam selected
r = radius defining the centroidal axis
y = distance from the fiber to the centroidal axis.

A creep strain-stress-time relation for simple tension and compression must now be selected. Many creep relations of this type have been proposed. In this paper, it will be assumed that the creep strain-stress-time relations for tension and compression are the same and are defied by

$$\epsilon_x = k_1\sigma_x^n + k_2(1 - e^{-qt})\sigma_x^n + B\sigma_x^n t \qquad (1)$$

or

$$\epsilon_x = k\sigma_x^n \qquad (2)$$

where

$$k = [k_1 + k_2(1 - e^{-qt}) + Bt] \qquad (3)$$

t = the time
σ_x = the stress
ϵ_x = the strain
q, k_1, k_2, B and q = material constants
e = the base for natural logarithms
k = the time function defined by Eq. 3.

The creep relation defined by Eq. 1 has been found to give a good approximation to test data (4). The representation of creep deformation by the three components corresponding to the three terms on the right side of Eq. 1 is also in agreement with physical concepts of creep as discussed by Orowan (5). Compared to other creep relations, Eq. 1 has the advantage that it leads to simpler mathematical formulations in the mechanics of creep. Because of the five experimental constants in Eq. 1, it also gives a better fit with test data than can be achieved with relations using fewer constants. Combining Eqs. *a* and 2

$$\sigma_x = \left[\left(\frac{y - \gamma}{r - y}\right)\left(\frac{\Delta d\phi}{d\phi}\right)\left(\frac{1}{k}\right)\right]^{1/n}. \qquad (b)$$

Considering equilibrium of forces on cross-sections in Fig. 3 produced by the moment M_ϕ,

$$\int \sigma_x da = 0, \quad \text{or,} \quad \int_{-h/2}^{h/2}\left[\left(\frac{y - \gamma}{r - y}\right)\left(\frac{\Delta d\phi}{d\phi}\right)\left(\frac{1}{k}\right)\right]^{1/n} bdy = 0 \qquad (c)$$

$$\int \sigma_z y da = M_\phi, \quad \text{or,} \quad \int_{-h/2}^{h/2} \left[\left(\frac{y-\gamma}{r-y} \right) \left(\frac{\Delta d\phi}{d\phi} \right) \left(\frac{1}{k} \right) \right]^{1/n} b y dy = M_\phi. \quad (d)$$

For a differential length $d\phi$ in Fig. 2, the resultant rotation due to the moments M_ϕ and $M_\phi' = N_\phi \gamma$ is

$$\Delta d\phi = \frac{(M_\phi - N_\phi \gamma)^n d\phi}{\int_{-h/2}^{h/2} \left(\frac{y-\gamma}{r-y} \right)^{1/n} \left(\frac{1}{k} \right)^{1/n} b y dy}. \quad (e)$$

To determine the statically determinate moment M_0, the condition will be used that there is no relative rotation of the cross-section AB with respect to cross-section EG (Fig. 2). That is

$$\int_0^{\pi/2} \Delta d\phi = \int_0^{\pi/2} (M_\phi - N_\phi \gamma)^n d\phi = 0. \quad (f)$$

In order to integrate Eq. f, M_ϕ and N_ϕ must be expressed in terms of ϕ. Considering the free-body diagram of section AOH of Fig. 2, as shown in Fig. 4, the summation of moments about section HI defines

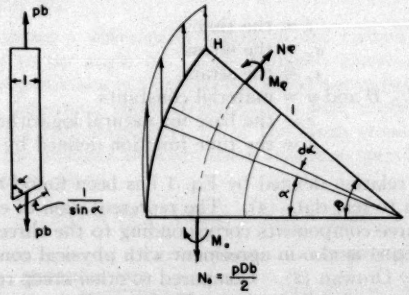


FIG. 4.

M_ϕ in terms of M_0. That is,

$$M_\phi = M_0 + \int_0^\phi \left\{ \frac{pDb}{2} \sin \alpha d\alpha \right\} \left\{ \frac{D}{2} \cos \alpha - \left(\frac{D}{2} - \frac{h}{2} \right) \cos\phi \right\} - \left[\left(\frac{pDb}{2} \right) \left(\frac{D}{2} - \frac{h}{2} \right) (1 - \cos \phi) \right] \quad (g)$$

or

$$M_\phi = M_0 + \left(\frac{pDb}{2} \right) \left(\frac{h}{2} - \frac{D}{4} \right) (1 - \cos^2 \phi). \quad (h)$$

The summation of the forces in the direction of the normal force N_ϕ gives

$$N_\phi = N_0 \cos^2 \phi \qquad (i)$$

where

$$N_0 = \frac{pDb}{2}. \qquad (4)$$

Placing values of M_ϕ and N_ϕ from Eqs. h and i in Eq. f,

$$\int_0^{\pi/2} \left[M_0 + N_0 \left(\frac{h}{2} - \frac{D}{4} \right) (1 - \cos^2 \phi) - N_0 \gamma \cos^2 \phi \right]^n d\phi = 0, \quad (5)$$

Eq. 5 could be used to determine M_0 provided the location of the neutral axis, as defined by γ, were known. The value of γ can be obtained by rewriting Eq. c as follows,

$$\int_{-h/2}^{h/2} \left(\frac{y - \gamma}{r - y} \right)^{1/n} dy = 0. \qquad (6)$$

Equation 6 can be solved for γ/r for different values of n and h/r. To determine γ/r, a series expansion method of evaluating the integral or a graphical method can be used. With γ known, Eq. 5 can be employed to evaluate the moment M_0. Equation 5 can be rewritten as

$$\int_0^{\pi/2} (A - B \cos^2 \phi)^n d\phi = 0 \qquad (j)$$

where

$$\left.\begin{aligned} A &= \frac{M_0}{N_0 r} + \left(\frac{h}{4r} - \frac{1}{2} \right) \\ B &= \frac{h}{4r} + \frac{\gamma}{r} - \frac{1}{2} \end{aligned}\right\}. \qquad (k)$$

Placing $\cos^2 \phi = (1 + \cos 2\phi)/2$ in Eq. j,

$$\int_0^{\pi/2} \left[\left(A - \frac{B}{2} \right) - \frac{B}{2} \cos 2\phi \right]^n d\phi = 0. \qquad (l)$$

Letting $2\phi = x$, $A - \frac{B}{2} = a$ and $-\frac{B}{2} = b$, Eq. l becomes,

$$\int_0^{\pi} (a + b \cos x)^n dx = 0$$

or

$$\frac{\pi}{2^n} \sum_{j=0}^{j=n/2} \frac{(-1)^j (2n - 2j)!\, a^{n-2j} (a^2 - b^2)^j}{j!(n - j)!(n - 2j)!} = 0. \qquad (m)$$

Equation m can now be solved for a for different values of n, h/r and $b = -B/2$. With values of a known, $A = B/2 + a$ can be determined and hence by Eq. k, $M_0/(N_0 r)$ evaluated. These calculations show that for all values of n,

$$\frac{M_0}{N_0 r} = \left(\frac{1}{4} + \frac{\gamma}{2r} - \frac{h}{8r}\right). \tag{7}$$

Equation 7 can now be used to determine M_0.

Determination of Stresses

With the statically indeterminate moment M_0 known, the stress σ_x can now be found. For this purpose, the equations of equilibrium (c) and (d) can be applied to the cross-section AB in Fig. 2, provided γ is replaced by some other quantity e_0. That is, for cross-section AB,

$$\int \sigma_x da = N_0 \quad \text{or} \quad \int_{-h/2}^{h/2} \left[\left\{\frac{y - e_0}{r - y}\right\}\left\{\frac{\Delta d\phi}{d\phi}\right\}\left(\frac{1}{k}\right)\right]^{1/n} b dy = N_0 \tag{n}$$

$$\int \sigma_x y da = M_0 \quad \text{or} \quad \int_{-h/2}^{h/2} \left[\left(\frac{y - e_0}{r - y}\right)\left(\frac{\Delta d\phi}{d\phi}\right)\left(\frac{1}{k}\right)\right]^{1/n} by dy = M_0. \tag{o}$$

Placing the value of $\Delta d\phi/d\phi$ from Eq. n in Eq. o,

$$N_0 \int_{-h/2}^{h/2} \left(\frac{y - e_0}{r - y}\right)^{1/n} y dy - M_0 \int_{-h/2}^{h/2} \left(\frac{y - e_0}{r - y}\right)^{1/n} dy = 0. \tag{8}$$

Equation 8 can be solved graphically for e_0/r for different values of n and h/r. Another method of finding e_0 is to evaluate the integrals in Eq. 8 by expanding the term $(1 - y/r)^{-1}$ in Eq. 8 into a series and selecting a few terms of this series.

With e_0 known, the unit rotation $\Delta d\phi/d\phi$ is completely defined by Eq. n or

$$\left[\frac{\Delta d\phi}{d\phi}\right]_{\phi=0} = \frac{N_\phi{}^n k}{\left[\int_{-h/2}^{h/2} \left(\frac{y - e_0}{r - y}\right)^{1/n} b dy\right]^n}. \tag{p}$$

The stress for any point on the cross-section AB can now be found from Eq. b, provided that in Eq. b, γ is replaced by e_0 and $\Delta d\phi/d\phi$ is replaced by its value given in Eq. p. Then

$$\sigma_x = \left(\frac{y - e_0}{r - y}\right)^{1/n} \left[\frac{N_0}{\int_{-h/2}^{h/2} \left(\frac{y - e_0}{r - y}\right)^{1/n} b dy}\right]. \tag{9}$$

The maximum stress for AB occurs at the inner edge B where $y = h/r$ and $r - y = a$. Then by Eq. 9, $\sigma = (\sigma_z)_{max}$, or,

$$\sigma = \left[\frac{\frac{h}{2} - e_0}{a}\right]^{1/n} \left[\frac{\frac{pDb}{2}}{\int_{-h/2}^{h/2} \left(\frac{y - e_0}{r - y}\right)^{1/n} b\,dy}\right], \tag{10}$$

where e_0 is evaluated by Eq. 8 and the integral in Eq. 10 can be found graphically or by expansion of the term under the integral into a series.

Equation 10 can be written as

$$\sigma = c\frac{F}{2br} \tag{11}$$

where

$$c = \frac{\left(\frac{h/2^n - e_0/n}{a}\right)^{1/n}}{\int_{-h/2}^{h/2} \left(\frac{y - e_0}{r - y}\right)^{1/n} dy}. \tag{12}$$

The nominal stress for the cross-section is

$$\sigma_N = \frac{F}{2bh}. \tag{13}$$

Then, by definition, the stress concentration factor is

$$f = \frac{\sigma}{\sigma_N} = c\left(\frac{h}{r}\right). \tag{14}$$

By means of Eqs. 12 and 14 values of f can be found for various values of h/r and n. Table I gives values of f for various representative values of h/r and n.

TABLE I.—*Stress Concentration Factors f.*

	Values of h/r		
n	1.5	1.0	0.5
1	2.60	2.55	3.85
3	1.68	1.87	2.14
5	1.44	1.39	2.68
7	1.34	1.30	1.19

CREEP STRAIN AT INNER EDGE OF HOLE

The creep strain at the inner edge of the hole can be found by placing the stress value defined by Eq. 10 in Eq. 2. Then

$$\epsilon = \frac{(kpDb)(h - 2e_0)}{4a\left[\int_{-h/2}^{h/2}\left(\frac{y - e_0}{r - y}\right)^{1/n} bdy\right]^n}. \tag{15}$$

Since the integral in Eq. 15 can be evaluated by graphical or other means, the creep strain can be obtained.

CONCLUSION

This paper develops an approximate theory for the creep stresses and strains in a plate with a circular hole subjected to an axial tensile load.

REFERENCES

(1) E. Z. STOWELL, "Stress and Strain Concentration at a Circular Hole in an Infinite Plate," NACA Technical Note No. 2073, April, 1950.

(2) H. F. HARDRATH AND L. OHMAN, "A Study of Elastic and Plastic Stress Concentration Factors Due to Notches and Fillets in Flat Plates," NACA Technical Note 2566, December, 1951.

(3) B. BUDIANSKY AND R. J. VIDENSEK, "Analysis of Stresses in the Plastic Range Around a Circular Hole in a Plate Subjected to Uniaxial Tension," NACA Technical Note No. 3542, October, 1955.

(4) J. E. GRIFFITH AND JOSEPH MARIN, "Creep Relaxation for Combined Stresses," *J. Mech. Phys. Solids*, Vol. 4, pp. 283–293 (1956).

(5) E. OROWAN, "Creep in Metallic and Non-metallic Materials," *Proc. First U. S. National Congress on Applied Mechanics*, published by ASME, March 1953, pp. 453–472.

V.2
195

THE FRANKLIN INSTITUTE

MEMBERSHIP

GOLD MEMBERS FOR 1959 HONORED

On May 20, 1959, the Institute paid tribute to four members who, in 1959, are celebrating fifty years of continuous membership. The new Gold Members are Theobald F. Clark, Walton Clark, Sr., James McGowan, Jr., and T. Edward Ross.

A reception honoring them and Fred H. Colvin, the senior Gold Member (class of '88), preceded the dinner and Stated Meeting. At the reception, the three new Gold Members present signed the Institute's Distinguished-Guest Book.

During the dinner, Wynn Laurence LePage, President, introduced the honored guests. At the Stated Meeting following the dinner, Morton Gibbons-Neff, Chairman of the Membership Committee, formally presented the four new Gold Members to the President, who congratulated them and gave them gold keys and gold membership cards. Judge Walton Clark, Sr., was unable to attend the festivities, so his key and membership card were given to his brother, Theobald F. Clark, another of the new Gold Members.

BIOGRAPHICAL SKETCHES OF THE NEW GOLD MEMBERS

THEOBALD F. CLARK

Theobald F. Clark of Chalfont, Pa., spent 43 years with the United Gas Improvement Company of Philadelphia. For 25 years, Mr. Clark was superintendent of Station A, 31st and Passyunk Ave., of the Gas Works Division. Before his retirement in January, 1957, Mr. Clark was Engineer of Construction for the Production Department.

Born November 29, 1889, in Chestnut Hill, Mr. Clark is the son of the late Dr. Walton and Louise Beauvais Clark. His father was President of The Franklin Institute from 1907 to 1923. Mr. Clark attended Chestnut Hill Academy. He graduated from Princeton University with a Civil Engineering degree in 1911 and from the University of Pennsylvania with a Mechanical Engineering degree in 1914.

He is a member of the Chalfont Burough Planning Commission, American Society of Mechanical Engineers, American Gas Association and the Pennsylvania Gas Association.

WALTON CLARK, SR.

Judge Walton Clark, Sr., of Darien, Conn., became the Presiding Judge of Darien's Probate Court in 1949. His previous positions ranged from timekeeper in the Klondike to attorney-at-law.

Judge Clark was born July 4, 1887, in Chicago. He is a brother of Theobald F. Clark. Judge Clark attended Philadelphia's Chestnut Hill Academy and the Central Manual Training School. He was graduated from Princeton University in 1908 and received his degree in law from the University of Pennsylvania in 1912.

Admitted to the Pennsylvania Bar in 1912, Judge Clark later was admitted to the bar in Oklahoma, New York and Connecticut. Judge Clark served as an attorney for several firms in Philadelphia, Tulsa and New York City, before opening his own practice in Darien, in 1946.

Judge Clark was an instructor at the Air-Mar Navigation Night School held at The Franklin Institute, 1943–1944, and a physics instructor at The Thomas School, Rowayton, Conn., 1947–1953.

JAMES MCGOWAN, JR.

James McGowan, Jr., of Beach Haven, N. J., a member of the Board of Managers of The Franklin Institute, has spent his 50-year career with the Campbell Soup Company. In 1908, he joined the company as a chemist. By 1946, he was President and in 1950 he was elected Chairman of the Board. Mr. McGowan retired in November, 1958.

Born February 8, 1886, in Milton of Campsie, Stirling, Scotland, Mr. McGowan was graduated from the Massachusetts Institute of Technology in 1908.

Mr. McGowan has served The Franklin Institute in various capacities during his 50-year membership. In addition to being a member of the Board of Managers, he presently is a member of the Membership Committee, Buildings and Grounds Committee and the Research and Development Committee.

Mr. McGowan serves either as trustee or member of the board for several educational and business organizations. In the past, he has served the Federal Government on various committees concerned with nutrition.

T. EDWARD ROSS

T. Edward Ross of Ardmore, Pa., has been a member of the American Institute of Certified Public Accountants for over a half century. He is a charter member of the Pennsylvania Institute of Certified Public Accountants.

Born November 18, 1887, in Northern Ireland, Mr. Ross attended school in Ontario, Canada, where his father was a school official.

He started his accounting career with John Heins in 1887. He became a partner of the firm Heins, Whelen, Lybrand & Co., in 1892. Six years later, he helped to found the firm of Lybrand, Ross Bros. & Montgomery, with which the still is affiliated.

Mr. Ross has served the Pennsylvania Institute of Certified Public Accountants in various official capacities including that of President. For many years, he has served the American Institute of Certified Public Accountants as chairman of the Committee on History.

V.2
195

At the 1959 Gold Member reception, T. Edward Ross signs the guest book. Watching (l. to r.) are Theobald F. Clark, Morton Gibbons-Neff, Fred H. Colvin and James McGowan, Jr.

NEW MEMBERS

The following elections were approved on June 4, 1959 by the Membership Committee by authority given to it by the Board of Managers.

ANNUAL

Harry V. Armitage
Souren Avakian
Colin Richard Barber
Frederic R. Benson
James E. Bentley
Arnold B. Blander
Walton Brand
G. Lupton Broomell, Jr.
Arthur J. Burke
Robert D. Carpenter
Ralph C. Carson, Jr.
R. J. Desenberg
Karl L. Dietrich
Andrzej Dobrowolski
Andrew E. Gal
Leo Goldstein
Theodore S. Gordon
Myron H. Halpern
Howard A. Harner, III
James I. Harper
Carl J. Hesch
Frank P. Houpt
Albert M. Kane
Lubomyr Kociuba
Dimitrios Kollias
Ernest Kurkjian
Louis F. Leidholdt
Alexander Lengyel
Edward R. Loveland
John J. Lotz
Winfield R. MacKeeby
Ronald H. Macpherson
Nicholas Maropis
Thomas J. Matcovich
J. S. McAdoo
Henry W. McCard
Charles F. McClurg
R. M. McCormick
J. David McKannan
Albert A. Melly
Paul D. Mercado
John H. Moore
Joseph F. Ridgway
William J. Southwell
Miller J. Sullivan
Manuel Tama
Mario Tama
John G. Thomas
John L. Thornley
Charles F. Wahlig
Leonard W. Williams

EDUCATOR

Samuel I. Askovitz
Robert D. Stout
William G. Wolfgang

NON-RESIDENT

Leroy Hooper
Martin Sandler

NECROLOGY

Archibald F. Borbeck '52
Alexander Henry, Jr. '53
William M. Huggins '55
Eugene J. Kopp '50
D. Edward McAllister '55
Carl F. Norberg '43
Charles M. Putney '42
Gustave T. Reich '51
Edward H. Reuss, Jr. '36
Henry N. Schramm '46
Ritner K. Walling '50
Daniel J. Walsh '51

LIBRARY

The Committee on Library desires to add to the collections any technical works that members may wish to contribute. Contributions will be gratefully acknowledged and placed in the Library. Duplicates received will be transferred to other libraries as gifts of the donor. 455 items have been added the past month.

Photostat service. Photostat prints of any material in the collections can be supplied on request.

The Library and reading room are open from 9 A.M. until 5 P.M. on Mondays, Tuesdays, Thursdays, and Fridays; 2 P.M. until 10 P.M. on Wednesdays; and from 9 A.M. until Noon on Saturdays.

RECENT ADDITIONS

AERONAUTICS

HOFF, NICHOLAS JOHN, ED. High Temperature Effects in Aircraft Structure. 1958.

LAMBERMONT, PAUL AND PIRIE, ANTHONY. Helicopters and Autogyros of the World. 1959.

MILNE-THOMSON, LOUIS MELVILLE. Theoretical Aerodynamics. Ed. 3. 1958.

SCHLICHTING, HERMANN AND TRUCKENBRODT, E. Aerodynamik des Flugzeuges. Vol. 1. 1959.

ARCHITECTURE AND BUILDING

BURKHARDT, CHARLES H. Residential and Commercial Air Conditioning. 1959.

ASTRONOMY

JARDETZKY, WENCESLAS S. Theories of Figures of Celestial Bodies. 1958.
ÖPIK, ERNST JULIUS. Physics of Meteor Flight in the Atmosphere. 1958.

ATOMIC ENERGY

LANSDELL, NORMAN. The Atom and the Energy Revolution. 1958.

BIOGRAPHY

HUNTER, DARD. My Life with Paper; an Autobiography. 1958.
WATSON-WATT, ROBERT ALEXANDER. The Pulse of Radar: the Autobiography of Sir Robert Watson-Watt. 1959.

CHEMISTRY AND CHEMICAL TECHNOLOGY

BAYER, ERNST. Gaschromatographie. 1959.
BREZINA, MIROSLAV AND ZUMAN, PETR. Polarography in Medicine, Biochemistry, and Pharmacy. 1958.
CHALLENGER, FREDERICK. Aspects of the Organic Chemistry of Sulphur. 1959.
DANIELLI, JAMES FREDERIC; PANKHURST, K. G. A. AND RIDDIFORD, A. C., ED. Surface Phenomena in Chemistry and Biology. 1958.
DIXON, W. T. AND FISHER, A. W. Chemical Engineering in Industry. 1958.
EVERS, NORMAN AND CALDWELL, DENNIS. The Chemistry of Drugs. Ed. 3. 1959.
GORDON, LOUIS; SALUTSKY, MURRELL L. AND WILLARD, HOBART H. Precipitation from Homogeneous Solution. 1959.
GRAHAM, RONALD POWELL AND CRAGG, LAURENCE HAROLD. The Essentials of Chemistry. 1959.
JONES, A. G. Analytical Chemistry. 1959.
NEBERGALL, WILLIAM HARRISON AND SCHMIDT, FREDERIC C. General Chemistry. 1959.
ROBERTS, JOHN D. Nuclear Magnetic Resonance; Applications to Organic Chemistry. 1959.
SHRADER, ROBERT L. Electronic Communication. 1959.
SOCIETY FOR ANALYTICAL CHEMISTRY. SCOTTISH SECTION. Proceedings of the Congress on Modern Analytical Chemistry in Industry . . . 1957. 1958.

COMPUTERS

MANDL, MATTHEW. Fundamentals of Digital Computers. 1958.
SMITH, CHARLES V. L. Electronic Digital Computers. 1959.

DICTIONARIES

WARREN, ALLAN. Dansk-Engelsk Teknisk Ordbog. Ed. 2. 1957.

ELECTRICITY AND ELECTRICAL ENGINEERING

AMERICAN INSTITUTE OF ELECTRICAL ENGINEERS. Proceedings of the Electronic Components Conference, 1950. 1958.
CARROLL, JOHN MILLAR. Modern Transistor Circuits. 1959.
ERICKSON, WILLIAM H. AND BRYANT, NELSON H. Electrical Engineering, Theory and Practice. Ed. 2. 1959.
HAMILTON, J. J. Reflex Klystrons. 1959.
INSTITUTE OF PRINTED CIRCUITS. How to Design and Specify Printed Circuits. n.d.
LOEB, LEONARD B. Static Electrification. 1958.
MITCHELL, F. H. Fundamentals of Electronics. 1951.
RHODES, DONALD R. Introduction to Monopulse. 1959.

RYDER, JOHN DOUGLAS. Electronic Fundamentals and Applications. Ed. 2. 1959.
SANDRETTO, PETER C. Electronic Avigation Engineering. 1958.
ZIMMERMAN, HENRY J. AND MATSON, SAMUEL J. Electronic Circuit Theory. 1959.

ENGINEERING

DILLIO, CHARLES C. AND NYE, EDWIN P. Thermal Engineering. 1959.
FINNIE, IAIN AND HELLER, WILLIAM R. Creep of Engineering Materials. 1959.
ORR, CLYDE AND DALLAVALLE, J. M. Fine Particle Measurement. 1959.

GEOLOGY

JOHANNSEN, ALBERT. Descriptive Petrography of the Igneous Rocks. Ed. 2. Vol. 1-4. 1958.

HISTORY

BURICKSON, SHERWIN. Concise Dictionary of Contemporary History. 1959.

INDUSTRIAL MANAGEMENT

BOWMAN, EDWARD H. AND FETTER, ROBERT B. Analysis of Industrial Operations. 1959.
MELMAN, SEYMOUR. Decision-making and Productivity. 1958.

INFORMATION STORAGE AND RETRIEVAL

CASEY, ROBERT SABERT AND OTHERS, ED. Punched Cards. Ed. 2. 1958.
PERRY, JAMES WHITNEY AND KENT, ALLEN. Tools for Machine Literature Searching. 1958.
INTERNATIONAL UNION OF PURE AND APPLIED CHEMISTRY. COMMISSION ON CODIFICATION, CIPHERING, AND PUNCHED CARD TECHNIQUES. A Proposed International Chemical Notation. 1958.

MANUFACTURE

BARNETT, ORVILLE T. Filler Metals for Joining. 1959.

MATHEMATICS

ANDREE, RICHARD VERNON. Programming the IBM 650 Magnetic Drum Computer and Data-Processing Machine. 1958.
BUSEMANN, HERBERT. Convex Surfaces. 1958.
Conference on Differential Equations. Proceedings; University of Maryland, 1955. 1956.
FULLER, GORDON. Plane Trigonometry. Ed. 2. 1959.
HALL, MARSHALL. The Theory of Groups. 1959.
LEWIN, L. Dilogarithms and Associated Functions. 1958.
LIPSON, H. AND TAYLOR, C. A. Fourier Transforms and X-ray Diffraction. 1958.
NOBEL, BENJAMIN. Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations. 1958.
PITT, H. R. Tauberian Theorems. 1958.

MECHANICAL ENGINEERING

JUDGE, ARTHUR W. Gas Turbines for Aircraft. 1959.
SLAYMAKER, ROBERT RIDGLEY. Mechanical Design and Analysis. 1959.

METALLURGY

AMERICAN SOCIETY FOR METALS. SOUTHWESTERN METAL CONGRESS. High Strength Steels for Aircraft. 1958.
GUY, ALBERT G. Elements of Physical Metallurgy. Ed. 2. 1959.
RAYNOR, GEOFFREY VINCENT. The Physical Metallurgy of Magnesium and Its Alloys. 1959.

NUCLEAR ENGINEERING

Handbook of the Atomic Energy Industry. 1958.
HOISINGTON, DAVID B. Nucleonics Fundamentals. 1959.

PETROLEUM TECHNOLOGY

HENGSTEBECK, R. J. Petroleum Processing. 1959.

PHOTOGRAPHY

HUNT, R. W. G. The Reproduction of Colour. 1957.
SCHWIDEFSKY, K. An Outline of Photogrammetry. 1959.

PHYSICS

CRANDALL, STEPHEN H., ED. Random Vibration. 1958.
CONSTANT, F. WOODBRIDGE. Theoretical Physics. 1954.
FANO, UGO AND FANO, L. Basic Physics of Atoms and Molecules. 1959.
HEISKANEN, WEIKKO ALEKSANTERI AND VENING MEINESZ, F. A. The Earth and Its Gravity. 1958.
HOLTON, GERALD JAMES AND ROLLER, DUANE H. D. Foundations of Modern Physical Science. 1958.
INTERNATIONAL SYMPOSIUM ON TRANSPORT PROCESSES IN STATISTICAL MECHANICS. BRUSSELS. 1956. Proceedings. 1958.
JELLEY, J. V. Čerenkov Radiation and its Applications. 1958.
KNAUSS, HAROLD P. Discovering Physics. 1951.
MELE, SILVIO. Le Basi della Meccanica. Vol. 1. 1955.
MERIAM, JAMES L. Mechanics. Ed. 2. Vol. 1-2. 1959.
PAI, SHIH-I. Introduction to the Theory of Compressible Flow. 1959.
PANOFSKY, WOLFGANG KURT HERMANN AND PHILLIPS, MELBA. Classical Electricity and Magnetism. 1955.
PAULI, WOLFGANG. Theory of Relativity. 1958.
RANDALL, ROBERT H. An Introduction to Acoustics. 1951.
RUECHARDT, EDUARD. Light; Visible and Invisible. 1958.

PLASTICS

NEUMANN, J. ALEX AND BOCKHOFF, FRANK J. Welding of Plastics. 1959.
PAQUIN, ALFRED MAX. Epoxyverbendungen und Epoxyharze. 1958.

SCIENCE

KEMENY, JOHN G. A Philosopher Looks at Science. 1959.
MUDRY, JOSEPH. Philosophy of Atomic Physics. 1958.
SARTON, GEORGE. A History of Science. 1959.

SUGAR

TIMOSHENKO, VLADIMIR PROKOPOVICH. The World's Sugar. 1957.

WOOD

UNITED NATIONS. FOOD AND AGRICULTURE ORGANIZATION. World Forest Products Statistics; a Ten-Year Summary, 1946–1955. 1958.

COMMITTEE ON SCIENCE AND THE ARTS

(Abstract of Proceedings of Stated Meeting held Wednesday, June 10, 1959.)

HALL OF THE COMMITTEE,
PHILADELPHIA, JUNE 10, 1959.

DR. WILLIAM G. SCHMIDT in the Chair.

The following reports were presented for final action:

No. 3335: Pottermeter.

This report recommended the award of an Edward Longstreth Medal to David M. Potter, Jr., of Union, New Jersey, "In consideration of the increased sensitivity and greater accuracy of an improved type of rotary flow meter developed by a unique application of the Venturi principle to balance the end thrust force to produce a free floating rotation free from the drag of end thrust."

No. 3359: Henderson Medal.

This report recommended the award of the George R. Henderson Medal to the Electro-Motive Division of the General Motors Corporation, of LaGrange, Illinois, "For the conception and demonstration of the feasibility of utilizing Diesel-electric locomotives on a wide scale in this country and for the design, development, standardization, production and servicing of these locomotives."

No. 3360: Clamer Medal.

This report recommended the award of the Francis J. Clamer Medal to Morris Cohen, of Cambridge, Massachusetts, "For his assiduous investigation and lucid exposition of the physical metallurgy of the heat treatment of steels as well as for a continuously high order of excellence of researches and publications covering the entire field of physical metallurgy."

No. 3361: Clark Medal.

This report recommended the award of the Walton Clark Medal to George G. Oberfell, of Bartlesville, Oklahoma, "In consideration of his significant contributions to the development of liquid petroleum (LP) gas as a source of domestic fuel."

D. S. FAHRNEY,
Secretary to Committee

BOOK REVIEWS

SWITCHING CIRCUITS (WITH COMPUTER APPLICATIONS), by Watts S. Humphrey, Jr. 264 pages, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1958. Price, $8.50.

The author's objective is to provide a graduate text and an aid to the practicing design engineer, particularly in the systems field, with a secondary objective of presenting enough theoretical material to enable the student to understand new methods as they become available.

Chapter 1, *Introduction*, describes the binary number-representation system, including square-rooting, with brief references to denary (base-10) and octonary (base-8) systems, and the binary representation of numbers in counters realized by transistor flip-flops.

Chapter 2, *Boolean Algebra*, presents the class calculus interpretation using the arithmetic symbolism with "+" as set union. The development begins with Venn diagrams and immediately introduces the concept of minterm and maxterm. It is unfortunate that these new expressions have been introduced since they are equivalent to the notion of atom and coatom already in use by Boolean algebraists. The laws of the class calculus are developed heuristically on the basis of the Venn diagrams rather than axiomatically, undoubtedly the best way for motivating the student.

Chapter 3, *Relay Control Networks*, considers combinational networks: "an algebraic expression can be written from a circuit and conversely a circuit can be constructed from an algebraic expression," and this fact is illustrated by examples showing that equivalent Boolean expressions correspond to equivalent series-parallel nets. The section on bridge networks gives examples of such nets, without, however, any algorithm for finding them or for verifying their design. A section entitled *Complementary Networks* considers topological duals that realize truth functions which are negations of each other, utilizing the transformation of a general four-node bridge. A useful ad hoc trick is given for realizing a three-terminal net such that for two terminals, one is always conducting to a common terminal, while the other is always open-circuited to common. The concluding section on relay trees scratches the surface of this important subject.

Chapter 4, *Coding*, considers a subject not properly part of switching theory, although it presents design problems to motivate the student thereof, especially in computer applications. The topics considered include permutations and combinations, probability, choice of base, binary-coded languages for base-10 (2 out of 5, XS-3, 8-4-2-1) and their error-detecting capabilities, mod-9 checks (the reviewer was scandalized to see the open arrow of implication used here) and parity checks and their error probabilities, error-correcting codes. The coverage is broad but not deep, and the selection of topics eminently sound.

Chapter 5, *Minimizing Aids*, deals with graphical methods for finding prime implicants, here called "factors," and disjunctive normal forms. Karnaugh maps are explained in terms of Euler-Venn diagrams and illustrated for 3 and 4 variables. Don't-care cases are considered and the process of finding prime implicates leading to a conjunctive normal form (referred to as factoring zeros and taking the negative). The reviewer feels that the explanation in terms of dual normal forms should have been mentioned also. The chapter concludes with an ingenious extension of the Karnaugh map to 5 and 6 variables, and a method of applying the distributive law to normal forms (called multilevel factoring by the author) by an inspection of the map—similar in spirit to finding prime implicants.

Chapter 6, *Circuit Logic*, considers several realistic design examples of combinational nets made up either of diode *and* and *or* circuits, or else of transistors. Only one type of transistor circuit is considered, that in which transistors are connected in series-parallel configurations and behave similarly to on-off relay contacts, except the control is via the base connection and the current flow is inherently unidirectional.

It is shown how diode gates can be cascaded, especially in two levels, to realize normal forms, and the concept of diode count is introduced.

The section on transistor logic requires more engineering knowledge of transistors than the sections on diodes and relays did of their physical counterparts. This seems unfortunate since transistor circuits are less well known than the others. The explanations are confusing because no polarities are shown on the diagrams and the signal states are not defined.

Chapter 7, *Boolean Matrices*, considers them as a shorthand for describing the connectivity of all terminal nodes of a general non-series-parallel controlled-contact network. A useful bag of tricks for simplifying such nets, centering around the techniques of node insertion and elimination, is presented. Some of these methods are basically wye-delta transformations.

Chapter 8, *Bilateral Networks*, commences by suggesting how to determine a lower bound to the number of springs required to realize a truth function by a controlled contact network and carries thru some syntheses using methods described in the preceding chapter. General networks of magnetic cores are then considered. Each core is viewed as a 2-terminal switching device that acts like a rectifier except the low-impedance direction is controlled by the orientation of core magnetization and hence ultimately by the control winding. This leads directly to nets which are amenable to treatment via matrices that are no longer symmetrical (off-diagonal terms occurring in codual pairs) and the previous matrix technique can be applied with minor changes. The combined effect of diodes and cores to eliminate sneak paths is also considered. This chapter contains much interesting original material.

Chapter 9, *Cascaded Networks*, covers the problem of designing iterative or recursive nets using chains of identical cells. Serial binary adders and modulo-3 remainder circuits are designed using relays, transistors, diode gating with and without clocking and inverting amplifiers. The comparative results serve to familiarize the reader with the characteristics of these different realization techniques.

Chapter 10, *Sequential Circuits*, develops the Mealy model and describes the reduction method for eliminating superfluous states in the state graph or diagram. Don't-care cases are included in the synthesis, which proceeds from verbal specification to state graph to reduced state graph and finally to the flip-flop assignment and the resulting combinational nets. The assignment problem is essentially a cut-and-try procedure and the author gives valuable suggestions for arriving at a reasonable solution. The reviewer feels that some attention should have been given to asynchronous relay circuits since he does not agree with the author's statement to the effect that there is no loss of generality in considering only synchronous circuits.

The following minor typographical defects were noted. Gray code is consistently misspelled except in the bibliography, and so is Semon. In the notation for the representation of a truth function by listing its atoms in denary notation, e.g. $\Sigma 2,3,5$, there is no indication of the number of variables. The upper matrix on page 195 has two misplaced ones in the upper left quadrant.

The author's class calculus interpretation of Boolean algebra is not consistent. The verbalisms are those of the statement calculus: "and," "or," "not," and the interpretation of the constants "a closed or short circuit is represented by 1, and an open circuit by 0," and of the variables "the expression xy can be used to represent a relay circuit," seems far afield from the class calculus. The reviewer prefers an all-out statement interpretation as for example in Kemeny, Snell, Thompson, *Introduction to Finite Mathematics*, principally because it is both more elementary and more comprehensive: "relay x is energized," "input y is low." The requirement that 1 represent a short circuit and "+" a parallel connection is dubious since this arbitrary polarization of the logic obscures the useful duality between open and short circuits and parallel and series connections. In the section on complementary networks of Chapter 4, almost all of the examples can be solved by inspection using topological duality. The question of notation is partly a matter of taste, but both logicians and mathematicians have abandoned "+" (except in Boolean rings), and as long as switching literature uses it, it is out of step with the basic sciences upon which it depends.

It would seem that more stress could have been placed on the truth table as the basic specification of a truth function, since this leads naturally to the definition of canonical

forms, Karnaugh maps, and description by listing atoms. The contributions of the logic school of switching theorists are slighted (for example, Quine is not even mentioned) and this works against achievement of the author's secondary objective, since new theoretical results are to be expected from this quarter. The de-emphasis of algorithms and proofs as contrasted with heuristic methods and convincement by example tends to unbalance the book. There is no mention of algorithms for finding prime implicants or the use of tables thereof, for finding output matrices by iterated squaring, for generating bridge networks from a tree expansion.

The text is a valuable addition to the literature, one that is especially noteworthy for its compactness despite a broad coverage of topics. It could be used successfully as an undergraduate as well as a graduate text in view of the fact that it contains selections from college algebra that reduce the prerequisites. Each chapter concludes with a carefully written summary and about twenty problems. The book shows every evidence of having been classroom tested, and this is particularly important to a prospective teacher.

As far as independent study is concerned, the book can be read profitably by those with a moderate amount of experience and even veterans will find it contains original contributions of interest to them.

GEORGE W. PATTERSON
University of Pennsylvania

ELECTRICAL MEASUREMENT ANALYSIS, by Ernest Frank. 443 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., 1959. Price, $8.75.

As implied by the inclusion of the word "analysis" in its title, Professor Frank's text departs from the familiar plan of most books on electrical measurements. This feature makes it a very welcome addition to the pedagogic literature. There is little discussion of the details of design and construction of specific manufacturers' instruments or components, but the fundamental physical laws of the operation of metering elements are not slighted in the least.

After a general chapter of introduction is placed a review of linear circuit analysis which is both compact and clear. For electrical engineering students this material may well be omitted, but for students of other engineering curricula, or for those studying on their own, this chapter is useful. Chapters 3 and 4 treat the basic metering unit, the galvanometer, with an excellent section on galvanometer dynamics.

The text then moves into material more pertinent to analysis by a study of the role and treatment of errors. The classes of systematic and random errors are defined and exemplified, the fundamentals of statistics and probability are presented, based on a frequency of occurrence definition. The criteria of precision discussed are limits of error, standard deviation and probable error. Methods for computing precision criteria for combinations of measurements, and of estimating the standard deviation of the mean are derived. Much of this is material which the undergraduate student sorely needs to develop an appreciation of the meaning of data, and which very often is not given him.

The author returns to more traditional electrical measurements material in succeeding chapters on deflection and null methods, excellent analyses of bridge methods, waveforms, and arc measurements. The final chapter returns to the analysis theme with its discussion of the treatment of data. Here we find a description of computing techniques, approximations, graphical representations and empirical equations, to aid the student in representing and interpreting his data.

There is no discussion of high frequency measurements, particularly in the case of distributed parameter circuits, such as waveguides. Measurements of frequency, time interval, and noise measurements are omitted. But these are really details, not fundamentals. The text has a fine set of problems with answers. Individual instructors may choose to alter the order of the material or the emphasis; nevertheless, the text should be easily adaptable to nearly all curriculum requirements.

A. C. BYERS
The Franklin Institute Laboratories

AIR POLLUTION CONTROL, by W. L. Faith. 259 pages, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $8.50.

Air Pollution Control is the latest addition to the technical bookshelf in an increasingly important field. The scope is wide and the

subject matter touches upon almost all related topics from meteorology to legal aspects. None of the subjects is developed deeply, but each is presented in an easy-to-read style. There is a minimum of mathematical and chemical formulary and controversial subjects have been avoided. The author has presented only the most accepted principles.

There are nine chapters: (1) The Air Pollution Problem; (2) Meteorology; (3) Smoke; (4) Dusts, Fumes, and Mists; (5) Gases; (6) Odors; (7) Automobile Exhaust; (8) The Air Pollution Survey; and (9) Legal Aspects.

The discussion of fuels in Chapter 3 is very good. There is also an excellent summary of combustion processes. The summary of laboratory and commercial collection systems in the chapter on Dusts, Fumes and Mists will be appreciated by the reader desiring a general description of these techniques. One of the better chapters describes the automobile exhaust problem. Exhaust composition as a function of the engine cycle, research and motor knock methods, and the effect of standard and automatic transmissions are included in this chapter.

Dr. Faith's presentation permits the inexperienced reader to obtain an easily assimilated background. The bibliography and manufacturer's references may then be used to obtain further detail. The experienced researcher or worker in this field will find very little that is new. The book will be of definite value, however, to the uninitiated or nonspecialist.

W. E. SCOTT
The Franklin Institute Laboratories

GROWTH AND PERFECTION OF CRYSTALS, edited by R. H. Doremus, B. W. Roberts and D. Turnbull. 609 pages, illustrations, 8½ × 11 in. New York, John Wiley & Sons, Inc.; London, Chapman and Hall; 1958. Price, $12.50.

This book is comprised of the papers presented at the International Conference on Crystal Growth held at Cooperstown, New York in August 1958.

This conference focused its attention on the fundamental mechanisms of crystal growth both in metallic and non-metallic systems. A fairly large number of papers deal with the growth and physical properties of "whiskers" (small filamentary single crystals). Studies dealing with the controlled growth and physical properties of these crystals have enabled investigations to partially unravel some of the mysteries of growth. There are many unsolved problems however; among these is the phenomenal strength of these small hair-like structures.

Six general sections contain some forty-odd articles and discussions. The first section is an introductory lecture by F. C. Frank whose pioneer work on the influence of dislocations on crystal growth stands as a beacon in the field. The next two sections deal with growth of whiskers and crystal imperfections. Here the papers deal with such pertinent subjects as dislocation studies, nucleation of ferromagnetic domains and elastic properties.

Growth studies involving solute and solvent phases are treated in sections four and five. The presentation covers a large range of materials. There are papers on liquid structure, salt dendrites, alloy crystal growth and the production of dislocation-free crystals. When one reads these chapters, he is immediately impressed with the great overlap, yet subtle differences in growth kinetics which are applicable when considering metallic versus nonmetallic systems.

The book contains much of the new important work on long-chained molecules. In support of this we find chapter six devoted entirely to the crystallization of polymers. The inclusion of papers on polymers promoted a useful exchange of views between scientists studying relatively simple molecules and those dealing with these more complex systems. This was indeed one of the aims of the conference.

The volume is well illustrated and contains some examples of very fine microphotography. There are author and subject indices and discussions are included in each section.

The book is highly recommended for physicists, chemists and metallurgists whose interest lies in the area of nucleation kinetics.

J. B. DREW
The Franklin Institute Laboratories

THE PHYSICAL METALLURGY OF MAGNESIUM AND ITS ALLOYS, edited by G. V. Raynor. 531 pages, diagrams, 5¼ × 8¼ in. New York, Pergamon Press, Inc., 1959. Price, $12.50.

This is Volume 5 in a series of Monographs on Metals Physics and Physical Metallurgy. Most texts devoted to a specific metal and its

alloys cover such points as mineral occurrence, beneficiation, foundry practice, fabrication metallography, etc. This book does not touch on these subjects but is devoted to the physics and physical metallurgy aspects of magnesium and its alloys.

Considerable attention has been focussed on magnesium in the last two decades. Primary interest has been directed toward its application as a light weight structural material for aircraft. However, its use has not been fully exploited because of its chemical reactivity. Research has resulted in many publications on the technology of magnesium and its alloys; but until publication of the present text, there has been no comprehensive review of the theoretical aspects of the subject.

The book opens with a brief introduction and a chapter on the physical and chemical properties of magnesium. This is followed by a detailed discussion on the electronic structure of solid magnesium. The author gives a review of the electron theory of metals sufficient to enable the reader to grasp the Brillouin zone structure. This section sets the pace for the remainder of the book which is strongly slanted in the direction of metals physics.

The general alloying behavior of magnesium is discussed in terms of valence and size factor effects. The author shows by numerous examples how theory can be used to explain in part the alloying behavior. He does an excellent job of showing how theory can also predict what elements might be usefully considered for the development of alloys for specific applications.

Over seventy pages are devoted to the important subject of intermediate phases in magnesium alloys with emphasis placed on the normal valency compounds and the Laves phases.

A chapter is devoted to the deformation characteristics of magnesium. Interpretation is made in terms of dislocation theory.

The latter half of the book is devoted to alloys of magnesium with various groups of the periodic table. Miscellaneous metals and gaseous elements are included. A great number of phase diagrams are given. There is an excellent closing summary of the influence of alloying.

Two appendices deal with the elastic constants, moduli, and a detailed treatment of the lattice spacings of magnesium alloys. There is both a subject and author index.

At times the book is difficult reading for the metallurgist not well grounded in the electron theory of metals. However, it is well worth his time and does much more than familiarize him with the metal physics of magnesium. It should convey to him a clearer understanding of how metals theory can be usefully applied to specific systems. The author has done an outstanding job in this respect.

R. L. SMITH
The Franklin Institute Laboratories

FOURIER TRANSFORMS AND X-RAY DIFFRACTION, by H. Lipson and C. A. Taylor. 76 pages, illustrations, 5½ × 8¼ in. London, G. Bell & Sons, Ltd., 1958. Price, $4.50 (available in U. S. from the Macmillan Co.).

Fourier transform techniques as applied to X-ray diffraction problems represent a very powerful tool. Unfortunately, the mathematical manipulations sometimes obscure the underlying physical principles. In this volume the authors have presented the fundamental techniques using the physical nature of the problem as a foundation. Mathematical vigor has been avoided, and therefore, the book can be read by anyone whose background in mathematics has not been too extensive.

The book begins by describing the subject briefly in the first chapter. From there the general expression for the Fourier transform of the electron density of a very small body scattering X-rays is derived. The same function is then re-derived for atoms, molecules and groups of molecules. The effect of symmetry operations on the transform are also presented. Crystalline diffraction is treated in Chapter 3, with discussions of structure factors and systematic missing reflections. A complete chapter is devoted to the important topic of convolutions and peak functions. A short, understandable derivation is given for the convolution with physical examples clearly presented. The peak function and its application also receive excellent discussions. This important chapter deserves special mention for its lucid discussion of the relation of the convolution and peak function to physical nature of X-ray diffraction by crystals. Chapter 5 contains the geometric fundamentals of the reciprocal and direct lattice with respect to X-ray diffrac-

V.2
195

tion. Succeeding chapters are devoted to the application of Fourier transforms to Fourier synthesis, structural analysis and defect structures. The greatest emphasis is on structural analysis.

In the Appendices, an example is given of a transform calculation, optical transforms, and finally, very neatly, shows some optical transforms as related to various topics in the main text.

The authors have presented an interesting subject in a very readable form.

E. KLOKHOLM
The Franklin Institute Laboratories

INTRODUCTION TO MONOPULSE, by Donald R. Rhodes. 119 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1959. Price, $6.00.

Donald Rhodes has prepared this monograph describing an important aspect of radar which may also be applied to certain other radiation problems. Dealing specifically with direction finding, it includes antennas and antenna systems as well as the circuits which utilize antenna signals. Early experiments were conducted as far back as 1928 by H. T. Friis, using a single antenna to determine the angle of arrival of atmospheric propagation. Refinements in direction finding have brought about the use of pairs of antennas in which the arriving signals could be compared electronically. The word "monopulse" comes from the theoretical concept that an angle of arrival could be determined from just one pulse returned from a passive source or transmitted from a beacon. This branch of radar direction finding, now termed monopulse, is a highly-developed field.

In this book the author attempts to give primarily theoretical concepts and outline the associated monopulse systems. Any reference to hardware or system components has been omitted. The work is largely descriptive with certain mathematical analyses included to cover general cases. Classification of all these systems according to accuracy and pitfalls which may be encountered has been one of the author's objectives.

This slim volume is, as its title indicates, introductory. However, through its study one could certainly become familiar with the language of monopulse systems and could derive an appreciation for the variety of system combinations. The author does not rely heavily upon many references but rather appears to derive his information from considerable personal experience. Of course, the usual references to the MIT Radiation Laboratory series and certain work done at Bell Laboratories are necessary.

The book will probably find a significant number of its users to be those experienced in some other branch of radar who are inclined to investigate monopulse as an alternate undertaking. It is hardly conceivable that this subject would be suitable for a course of instruction at the university level, because it is rather brief and somewhat specialized. The author's style is clear and understandable, so that a practicing engineer should have little trouble covering this text in several weeks' time. It is also likely that the reader would wish to supplement this study by frequent reference to other books on the particulars of radar and electromagnetic radiation.

C. W. HARGENS
The Franklin Institute Laboratories

BOOK NOTES

FUNDAMENTAL ASPECTS OF REACTOR SHIELDING, by Herbert Goldstein. 416 pages, diagrams, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1959. Price, $9.50.

This book is based on a report prepared by the author for the Atomic Energy Commission. Concentrating on a discussion of the fundamentals of reactor and shield design, the book's twofold aim is to encourage further investigations into the basic physics of shields and to supply an exposition of the shielding processes. A certain background is assumed in radiation biology and nuclear physics on the part of the reader, as they enter into the subject of shielding, but full discussion of them is not given.

INFORMATION THEORY AND STATISTICS, by Solomon Kullback. 395 pages, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $12.50.

Using a general probabilistic setting, this book describes the logarithmic measures of information and their applications to the testing of statistical hypotheses. The author unifies his applications by consistent use of the concepts and properties of information theory. These applications are to the analysis of samples of fixed size, from multinomial populations and contingency tables (nonparametric), and parametric problems about the exponential family of populations, particularly the multinomial, Poisson, and normal. The discussions provide the foundation for an extension to more complex stochastic processes, like sequential analysis and those of communication theory. Numerous worked examples are included in this volume.

APPLIED PETROLEUM RESERVOIR ENGINEERING, by B. C. Craft and Murray F. Hawkins, Jr. 437 pages, diagrams, 6 × 9 in. New York, Prentice-Hall, Inc., 1959. Price, $12.00.

The science of primary and secondary recovery of oil and gas is treated competently in this original text. The extraordinary growth of the petroleum industry in recent years has been largely responsible for the improved methods for analysis and prediction of reservoir and well performance. This text presents these new methods, relating theory to practice, with a field engineering approach. Suitable for use as a first course in reservoir engineering, the text provides numerous examples and problems with answers.

THE STRUCTURE OF ELECTROLYTIC SOLUTIONS, edited by Walter J. Hamer. 441 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $18.50.

The latest volume in the Electrochemical Society Series, this book is based on a symposium held in Washington in May 1957. Twenty-seven chapters, written by authorities from eight countries, cover all phases of current work on electrolytic solutions. In addition to properties of electrolytic solutions, the chapters cover dilute and concentrated solutions, fused electrolytes and the pure ionic solid state. New experimental techniques are described in several chapters. Complex ions, ion pairs, polyelectrolytes, fused salts, ionic crystals and the rare earths occupy a good share of the text. Each author uses his own nomenclature and symbols; the editor did not attempt to standardize these, since the main purpose of the symposium was to stimulate thought and future work in the field. The references appended to each paper include historical sources as well as recent work up to and, in some cases, including 1958.

INDUSTRIAL FATTY ACIDS, edited by E. Scott Pattison. 230 pages, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1959. Price, $7.00.

Reflecting the growing industrial importance of fatty acids derived from tall oil, one of the most important recent advances, this book covers the production and processing of fatty acids with emphasis on practical technology and the chemistry involved with it.

V. 19

The many industries using fatty acids will find here a wealth of dependable information on these materials.

RESEARCHES IN GEOCHEMISTRY, edited by Philip Hauge Abelson. 511 pages, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $11.00.

This volume provides a comprehensive view of current research in geochemistry. Written by authorities in their respective fields, each chapter reviews briefly the present status of research in the area of inquiry and describes in detail the authors' recent contributions. Each chapter contains an extensive selected bibliography. Some of the major contributions in this text include the description of techniques used to estimate temperatures of formation of numerous ore deposits; dating methods for the study of precambrian stratigraphy, already successfully employed to date events occurring early in the earth's history; discussion of the use of tritium. A valuable reference source for graduate students and all who are interested and active in earth science.

SYMBOLIC LOGIC AND INTELLIGENT MACHINES, by Edmund C. Berkeley. 203 pages, diagrams, 6 × 9 in. New York, Reinhold Publishing Corp., 1959. Price, $6.50.

The role of symbolic logic in the programming of automatic computers and such complex devices as robots is discussed in this informative text. The principles, methods and purposes of symbolic logic and Boolean algebra, on which the programming of intelligent machines is based, are all explained in a practical manner by the author. The design and principles of both small and large machines are included, along with specific problems that confront these machines.

TRANSISTORS, by Angelo C. Gillie. 262 pages, diagrams, 6 × 9 in. New York, Prentice-Hall, Inc., 1959. Price, $7.95.

As transistors are being universally adopted and recognized as a replacement for vacuum tubes, this new text provides an understanding of the behavior of transistors in various applications, without the usual prerequisite of advanced mathematics. The principles can be easily understood by anyone familiar with Ohm's Law, a-c. theory and fundamental algebraic processes. Organization of the material is simple and logical, from general linear and non-linear conductors, to linear resistive control devices, to the transistor—and finally to its application. The author has provided, at the end of each chapter, an abundance of questions and has included 200 problems with answers to augment and intensify the understanding of theoretical principles.

THE UPPER ATMOSPHERE, by H. S. W. Massey and R. L. F. Boyd. 333 pages, diagrams, 6 × 9 in. New York, Philosophical Library, 1959. Price, $17.50.

This volume is an account of the phenomena in the upper atmosphere studied during the International Geophysical Year. Methods of interpreting the coded signals of rockets and satellites and of finding their positions and speeds are fully described. Radio fadeout, radio absorption, night airglow, aurorae, meteors, cosmic rays and currents responsible for magnetic variations are among the phenomena receiving detailed attention. Higher mathematics are not required by the reader and some chapters succeed in avoiding mathematical exposition altogether. This is primarily a book for students, or for scientists in other fields who require a reliable outline of the subject of space travel.

HIGH ALTITUDE AND SATELLITE ROCKETS, a symposium. 136 pages, illustrations, 8½ × 11 in. New York, Philosophical Library, 1959. Price, $15.00.

These are the proceedings of the first symposium on high altitude and satellite rockets, held in Great Britain in July 1957. The twelve papers presented by British and American authors deal with some of the design and propulsion problems of high altitude rockets, recovery after re-entry, high temperature materials, instrumentation, telemetry and guidance and some of the problems of human ascent beyond the atmosphere.

METHODS BASED ON THE WIENER-HOPF TECHNIQUE FOR THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS, by B. Noble. 246 pages, diagrams, 5½ × 8½ in. New York, Pergamon Press, 1958. Price, $10.00.

Published as Volume 7 in the International Series of Monographs on Pure and Applied

Mathematics, this book describes methods based on the Wiener-Hopf technique for solving certain boundary-value problems involving partial differential equations. Prerequisite for the material is a knowledge of the Laplace transform, its complex inversion formula, and integration in the complex plane. The author discusses the complex variable in Fourier transforms, half-plane problems, and wave problems. Extensions and limitations of the Wiener-Hopf method include simultaneous Wiener-Hopf equations and Laplace's equation in polar coordinates. In Chapter 5 on Some Approximate Methods, the author treats some interesting problems which cannot be solved exactly. A six-page bibliography of related references includes 1958 references.

STATICALLY INDETERMINATE STRUCTURES, by Jack R. Benjamin. 350 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1959. Price, $11.00.

Design as a supplement to standard texts on analysis and design, this book covers the approximate analysis of statically indeterminate structures using statics and the principles of deflected structures. There are many detailed sketches on the design of deflected structures. Approximate analysis techniques are given for continuous beams, rectangular rigid frames with vertical loads and with lateral loads, shear-wall structures with lateral loads, bents, arches, and bridges. The final chapter relates analysis to design.

SILICONES, by Robert N. Meals and Frederick M. Lewis. 267 pages, illustrations, 5 × 7½ in. New York, Reinhold Publishing Corp., 1959. Price, $5.95.

The scope of this text includes the manufacture, properties and applications of the silicones, with data on the properties of silicone resins, fluids and rubbers at high temperatures. It was written especially for design engineers, formulators and manufacturers in all industries having interest in silicones as engineering and raw materials. Containing many case histories of present industrial applications, including the uses of silicones as chemicals, rubbers, resins, fluids, greases, adhesives, sealants, and surface coatings, this handbook presents a concentrated, readable summary of the subject.

CELESTIAL MECHANICS, by E. Finlay-Freundlich. 150 pages, diagrams, 5½ × 8½ in. New York, Pergamon Press, 1958. Price, $7.50.

This volume deals with one problem of astronomy, that of the motion of celestial bodies in the gravitational field of the sun. Reinterest in this problem has increased tremendously since the successful creation of artificial satellites. Using straightforward mathematics, the book covers a wide field and will be a valuable aid to all students of astronomy, physics or mathematics.

THE POTENTIAL THEORY OF UNSTEADY SUPERSONIC FLOW, by John W. Miles. 220 pages, diagrams, 5½ × 8½ in. New York, Cambridge University Press, 1959. Price, $8.50.

In this monograph, the author gives a systematic and comprehensive survey of the theory of those aerodynamic forces which result from unsteady motion of the structural components of high speed aircraft. Beginning with a full discussion of the basic equations of potential flow in their exact and approximate forms, the available methods of solution are developed and applied to typical supersonic wings, slender bodies and wing-body combinations. Illustrative calculations are given for harmonic motions (such as occur in dynamic stability and flutter problems) and also for transient motions. An appendix is provided dealing with reverse flow theorems, and a bibliography containing over 300 references. This is a book especially valuable to those engaged in research into unsteady flow, and for engineers concerned with practical flutter and stability analysis.

THE BIRTH OF THE STEAMBOAT, by H. Philip Spratt. 149 pages, plates, 5½ × 7½ in. London, Charles Griffin & Co., Ltd., 1958. Price, 28s.

This is an historical account of the ideas, experimentation and persistent efforts which culminated in the achievement of the invention of the steamboat some 150 years ago. The author traces the development in preceding centuries of the basic concepts of steamship propulsion which led to eventual success. It also gives some account of the manifold difficulties faced and overcome by the pioneer inventors and designers. Making a genuine

contribution to scientific history, this text breaks new ground since no work of just this scope has yet been attempted, being copiously documented from original research.

INTRODUCTION TO THE THEORY OF COMPRESSIBLE FLOW, by Shih-I Pai. 385 pages, diagrams, 6 × 9 in. Princeton, D. Van Nostrand Co., Inc., 1959. Price, $9.75.

In this volume the author briefly reviews the physical properties of gases and then discusses in detail the theory of compressible flow of inviscid fluids, covering both steady and non-steady flow. Such advanced topics as transonic flow, hypersonic flow, and rotational flow are thoroughly explored. Briefly discussed also are shock waves in a real gas and diabetic flow. Three chapters are devoted to an accurate and detailed summary of available information on the most important interaction phenomena, chemical reaction and the electromagnetic forces in the flow field (magnetogasdynamics). This book provides a thorough treatment of the subject of compressible flow hoping to further independent research in the field.

BOUNDARY LAYER RESEARCH, edited by H. Görtler. 411 pages, illustrations, 6¼ × 9¼ in. Berlin, Springer-Verlag, 1958. Price, DM 67.50.

In August, 1957, the International Union of Theoretical and Applied Mechanics sponsored a symposium on boundary layer research, in Freiburg, Germany. The present work encompasses all the lectures presented at that symposium. Participants came from seventeen nations and their contributions are presented in one of three languages—English, German or French. Written discussions follow each of the eight sections. There are thirty-two papers covering fundamental problems in boundary layer research, particularly with reference to aeronautics. Even though the specialists may not be able to read all three languages, they will find many challenging ideas presented in whatever languages they can read.

YOU CAN ADVANCE SCIENCE EDUCATION

Today, more than ever before in its 135-year history, there is vital need for The Franklin Institute to effectively promote education in science and technology. It is imperative that we meet this challenge by providing adequate educational opportunities in these fields. This requires vision, objective planning, and money. We have more than enough of the first two requisites, but far too little of the third.

Our programs are aimed at professional scientists and industry, as well as the lay public and young people seeking inspiration and guidance in choosing a career. The Institute's educational programs are impressive, for they begin with students in the early grades of our elementary schools and continue throughout an individual's professional or industrial life. With more funds at our disposal, the scope and vigor of these activities could be greatly increased.

The Franklin Institute is not richly endowed. It is a non-profit organization, depending for encouragement and support on an understanding public. Capable and conservative management assures wise administration of all funds.

Your gift or bequest, large or small, will be deeply appreciated and will be used effectively to broaden the Institute's educational usefulness. There is a warm satisfaction in giving financial support to an organization that has pioneered in, and is dedicated to, the advancement of science and technology through education.

When property is transferred, title should be in the name of The Franklin Institute of the State of Pennsylvania for the Promotion of the Mechanic Arts.

The Secretary of The Franklin Institute will gladly furnish you with additional information. Write to him at The Franklin Institute, Benjamin Franklin Parkway at Twentieth Street, Philadelphia 3, Pennsylvania.

PUBLICATIONS RECEIVED

The Editors wish to call attention to these new books which have been received in the JOURNAL office for review. Lack of space prevents publication of full reviews.

SUCCESSFUL TECHNICAL WRITING, by Tyler G. Hicks. 294 pages, 5½ × 8 in. New York, McGraw-Hill Book Co., Inc., 1959. Price, $5.50.

PROGRESS IN NUCLEAR ENERGY. LAW & ADMINISTRATION, VOLS. 1-2, edited by Herbert S. Marks. 994 pages, 6 × 9 in. New York, Pergamon Press, 1959. Price, $26.50.

THE PHYSICS OF ELECTRICITY AND MAGNETISM, by William Taussig Scott. 635 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $8.75.

THERMODYNAMICS, by Gordon J. Van Wylen. 567 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1959. Price, $7.95.

CALCULUS WITH ANALYTIC GEOMETRY, by Angus E. Taylor. 762 pages, diagrams, 6 × 9 in. New York, Prentice-Hall, Inc., 1959. Price, $8.50.

A CONCISE ENCYCLOPEDIA OF WORLD TIMBERS, by F. H. Titmuss. Second edition, 264 pages, plates, 6 × 9¾ in. New York, Philosophical Library, 1959. Price, $15.00.

ELEMENTS OF WAVE MECHANICS, by N. F. Mott. Student edition, 156 pages, diagrams, 5½ × 8½ in. New York, Cambridge University Press, 1959. Price, $2.95 (paper).

PRINCIPLES OF COLLEGE PHYSICS, by George Shortley and Dudley Williams. 898 pages, diagrams, 6 × 9 in. New York, Prentice-Hall, Inc., 1959. Price, $9.00.

THE PHYSICAL SCIENCES, by E. J. Cable, R. W. Getchell, W. H. Kadesch, W. J. Poppy and H. E. Crull. Fourth edition, 553 pages, illustrations, 7 × 9¾ in. New York, Prentice-Hall, Inc., 1959. Price, $6.95.

NATURAL SELECTION AND HEREDITY, by P. M. Sheppard. 212 pages, diagrams, 5½ × 8½ in. New York, Philosophical Library, 1959. Price, $6.00.

ENCYCLOPEDIA OF CHEMICAL REACTIONS, VOL. VIII, compiled by C. A. Jacobson and edited by Clifford A. Hampel. 533 pages, 6 × 9 in. New York, Reinhold Publishing Corp., 1959. Price, $14.00.

PLANT PROPAGATION, by Hudson T. Hartmann and Dale E. Kester. 559 pages, illustrations, 6 × 9 in. New York, Prentice-Hall, Inc., 1959. Price, $8.75.

F. L. STILLWELL ANNIVERSARY VOLUME. 302 pages, illustrations, plates, 5½ × 8½ in. Melbourne (Australia), The Australasian Inst. of Mining & Metallurgy, 1958. Price, not given.

TRIGONOMETRIC SERIES, Vols. I and II, by A. Zygmund. Second edition, 7 × 10 in. Vol. I, 383 pages; Vol. II, 354 pages. New York, Cambridge University Press, 1959. Price, $15.00 each or $27.50 a set.

CURRENT TOPICS

Ultraviolet Light Absorber.—Polyethylene and polypropylene plastic products for outdoor use soon will be economical and practical because of a new ultraviolet light basorber developed by American Cyanamid Company. Known as UV 314, the new absorber is expected to be available commercially earlynext year.

A major breakthrough in ultraviolet absorber technology, the new product, 2,2′-dihydroxy-2-n-octoxybenzophenone, is expected to increase annual polyethylene consumption as much as 10 to 50 per cent, according to Dr. J. A. Weicksel, product supervisor for Cyanamid's line of Cyasorb ultraviolet absorbers. Polyethylene consumption in 1958 reached 832 million pounds. A similar impact on the polypropylene market is anticipated.

"UV 314 will be used initially by the plastics industry in polyethylene fibers, films and sheeting," Dr. Weicksel predicted. "Its greatest potential," he continued, "will be in applications for which polyethylene, because of its ultraviolet instability, has not yet been considered."

Marine ropes and fabrics for automotive upholstery and lawn furniture are some of the products that may be made with light-stabilized polyethylene fibers. Typical film and sheeting applications include mulching and greenhouse sheeting, tarpaulins, high altitude balloons and moisture barriers for building construction.

Ultraviolet instability has limited polyethylene's use in outdoor applications, Dr. Weicksel pointed out. When exposed to weather and sunlight for only a month, it loses its flexibility and rapidly becomes stiff and brittle.

Today's commercially available ultraviolet absorbers, with the exception of carbon black, have not been compatible with polyethylene. Although extremely effective as a light absorber, carbon black limits the use of the resin to opaque, black products.

Cyanamid's UV 314 absorber has been tested extensively by the company under both accelerated and outdoor exposure conditions. In tests conducted in Arizona sunlight, ten-mil polyethylene sheets containing 0.171 per cent UV 314 retained 85 per cent elongation after two months. Untreated sheets retained only 9 per cent after two months.

To learn how the industry felt about the effectiveness of UV 314, Cyanamid, not a producer of polyethylene, submitted samples of the new material to manufacturers late last year. Preliminary reports on their exposure tests are confirming the company's results.

UV 314 is a pale yellow powder that absorbs strongly in the 300–375 millimicron range of the ultraviolet. Its long alkyl group (n-octyl) at the ether linkage makes it compatible with polyolefins.

Samples of the new absorber are limited by Cyanamid at the present time to polyethylene manufacturers. Process development material should be generally available in August.

Injectable Form of Chloromycetin.—Chloromycetin Succinate, a new injectable form of the broad-spectrum antibiotic chloramphenicol, has

been introduced by Parke, Davis & Company.

Studies with the new compound—the sodium salt of the monosuccinate ester of chloramphenicol—indicate it is rapidly absorbed by the patient, produces effective serum levels quickly and has the virtues of simplicity and ease of administration, the company said. In addition, Chloromycetin Succinate is "better tolerated at the site of injection than are some similarly administered antibiotics."

Chloromycetin Succinate (chloramphenicol sodium succinate) exerts the same broad coverage against disease-producing organisms as produced by other forms of Chloromycetin products. The antimicrobial spectrum of chloramphenicol is well established and includes susceptible streptococci, staphylococci, gonococci, pneumococci, brucellae, salmonellae, shigellae, and many other gram-positive and gram-negative organisms, as well as certain rickettsiae and viruses, the company said.

Chloromycetin Succinate, which can be administered intramuscularly, intravenously or subcutaneously, is supplied in powdered form in vials for dilution before injection.

Computer Translation into Braille.—Translation of printed text into braille at electronic speeds has been achieved with an International Business Machines Corp. electronic computer.

The procedure creates in minutes a raised braille printing plate suitable for production of books for the nation's 350,000 sightless persons. It was developed by IBM mathematicians working in conjunction with the American Printing House for the Blind, leading braille book publisher.

The English text-to-braille computer translation program is expected to be put to work immediately to make up for the serious shortage of qualified braille translators, whose training normally takes two years. A knowledge of braille by computer personnel is unnecessary.

The new method of creating braille was developed for a standard IBM 704 data processing system, a powerful computer which has been installed in many industries to handle complex scientific and engineering problems. The 704 can translate a 300-page book in an hour—a job which would take more than six days for a skilled translator.

Writing braille requires knowledge of many complicated rules of usage. Symbols often change their meaning in different contexts. Braille consists of 63 combinations of 6 raised dots to represent not only the alphabet, numerals and punctuation marks, but also 183 special contractions and abbreviations, similar to written shorthand.

Texts to be translated by the computer are first transferred to punched cards by means of a key punch with a conventional typewriter keyboard. The cards are then fed directly into the computer, which has had previously stored in its memory a program or set of rules for conversion into braille. The machine executes as many as 600 instructions per word in less than the fortieth part of a second. The computer determines contractions and abbreviations by matching against an alphabetical table of braille equivalents.

The translated text emerges from the computer in coded symbols on punched cards, and these, in turn, are fed to a printer which reproduces the braille symbols above the English text for editing purposes. After editing, the corrected deck of punched cards is fed to an embossing machine which

creates metal plates for use in a rotary press.

The present program, which required a year for completion, was the work of the Mathematics and Applications Department of IBM's Data Processing Division headed by Dr. Joseph E. Flanagan. This group has been exploring the feasibility of proposed applications of computers ranging from weather forecasting to job shop simulation.

This new technique is expected to be used for the translation of textbooks and technical titles now not available in braille to broaden career opportunities for the nation's blind.

Static Protective Relay.—A new static protective relay capable of operating at speeds faster than conventional electro-mechanical designs has been developed by General Electric's Low Voltage Switchgear Department, in Philadelphia.

Development of the new static phase comparison relay—first in General Electric's new line of static relays for wide application in the protection of high voltage transmission lines and power system components—opens a new era in the art of protective relaying, according to department general manager Frank E. Stehlik.

Believed to be the most significant advancement in protective relaying since the introduction of high-speed relays thirty years ago, the new transistorized design has completed exhaustive field tests on the Muskingum-Philo 138 KV line of The Ohio Power Company.

Operating within one and one-half cycles, the new static relay greatly reduces damage caused by line faults and improves stability to give better system protection.

Designed for use with General Electric's new transistorized carrier current equipment, the relay provides high-speed simultaneous tripping of transmission line breakers for all internal faults by comparing phase relationships of currents entering and leaving a line section.

The transistorized units can be widely applied in the protection of high voltage transmission lines.

Design flexibility permits use of the relay on multi-terminal transmission lines. Addition of new terminals to existing lines is possible without major changes to the relaying system. Because of its high sensitivity, the new static protective relay detects lower magnitude faults and permits heavier line loading with an equivalent protection level.

Incorporating static components, transistors, stabistors, diodes, and reference diodes, the static phase comparison relay eliminates the necessity of tubes, and their related maintenance costs.

All transistor circuits are equipped with positive temperature stabilization to give consistent performance over a wide range of temperatures.

A "plug-in" modular construction facilitates testing and maintenance.

Weather Display Device.—An automatic printing device which can display a new message on local airfield weather observation in 1.2 seconds has been developed for U. S. Air Force testing by Burroughs Corporation, Detroit.

The machine, designed to print out weather reports in plain language and display these reports in such strategic locations as the control tower, pilot ready room, Ground Control Approach station, operations office and weather offices, will be tested to determine its applicability to the 433L program, the Weather Observing and Forecasting

System now being developed by the Air Force.

Called Burroughs Display Device S401, the machine receives and prints by electrostatic process incoming messages on paper tape from data received via standard telephone party lines. It will also print from digital signals received from computers, punched paper tape, electronic message storing devices or magnetic drums, Burroughs engineers declared. Up to 12 of these devices may be operated across the same telephone line.

Characters, $\frac{5}{8}$ in. high by $\frac{3}{8}$ in. wide, can be viewed readily in a dimly lighted room at distances up to 12 feet.

Airfield weather reports on temperature, visibility, wind direction and velocity, barometric pressure and humidity automatically will be fed into an electronic message assembler which, in turn, will transmit the information by code to the S401. The S401 will convert the code into plain language on the display tape.

The equipment is designed so that test data will change with every significant change in the weather, enabling users to keep aware of even the smallest important weather change.

The Air Force expects the electrostatic display device to go on line testing within 18 months.

Molecular Electronics.—A $2-million development contract in the field of "molecular electronics" has been awarded to Westinghouse Electric Corporation. The contract was issued by ARDC's Wright Air Development Center as part of a broad program effort in this new electronic area. The development centers on a revolutionary method of "growing" germanium crystals as thin, uniform flat ribbons. Formerly, the raw product was round and about the size of a .50 calibre bullet. Tedious slicing and polishing of the material then followed.

Research now shows the material may be literally "grown" as thin, uniform flat ribbons.

Research results to date indicate that the new method may lead to the development of outer-space electronic equipment 1,000 times smaller and lighter than anything now in existence.

Carrying this new concept through to its conclusion and to the development of an over-all system, it would not be hard to foresee replacement of an electronic system the size of a breadbox with one the size of a cube of sugar. Consequently, the promise of drastic reductions in size, weight and power requirements for electronic equipment could allow space vehicles to perform a greater number and wider range of functions.

Explaining this new germanium growing technique, scientists of the Westinghouse Research Laboratories said the material grows directly in the exact form in which this semiconducting material is used for practical purposes in transistors and similar devices.

Dr. S. W. Herwald, Westinghouse vice president, Research, described the "molecular engineering" concept as a "basically new approach to the building of electronic systems through the use of our new knowledge of the structure of materials." He said it deals with dimensions comparable to molecules themselves and "is not just an improved method of packaging present-day components with already developed philosophies."

"Dendritic germanium has broadened our thinking about the whole technology of solid state devices of the future," Dr. Herwald said. "One can envision, for example, the process at work in a machine that continuously, automatically, and at high speed turns out finished transistors directly from an input of raw germanium and the

two or three other materials required to put a transistor into final form."

In use, these very small pieces of crystal could perform the functions that now require as many as a dozen present-day components using conventional circuitry. Two possible functions of the minute equipment are the sensing of environmental conditions and transmitting the data to any desired location. Feasibility studies are being conducted on infrared, reconnaissance, flight control, communications, and other military applications.

As an example of the size and weight reductions possible under the new concept, the most modern system used today to measure the intensity of light in space has a volume of about one cubic inch and a weight of approximately seven grams. The Air Force has conducted demonstrations with a Westinghouse light telemetry subsystem which has a volume of one one-thousandth of a cubic inch and a total weight of two one-hundredths of a gram. This equipment not only measures change in light intensity but also produces a signal capable of transmission to relate the degree of change in intensity.

The new technique for growing germanium was the work of two Westinghouse research physicists, Dr. R. L. Longini and Dr. A. I. Bennett, both of the laboratories' solid state physics department.

"The ease of processing dendritic germanium into usable form contrasts sharply with present practices," the Westinghouse scientists said. "To process conventional germanium ingots into useful form, they must be sliced into thin wafers somewhat like a loaf of bread is cut into slices. Because germanium is so hard and brittle, the cutting is done with a diamond-tipped saw, and each slice has to be three to five times the desired thickness simply to keep it from shattering.

"The slices must then be ground down to the required thickness, further cut into small squares, and finally polished. Only then is the germanium ready to be fashioned into finished transistors or other devices. As a result of all this cutting and polishing, about 80 per cent of the original ingot is thrown away as germanium 'sawdust.'

"Not only is dendritic germanium of the proper thickness for direct use, its ultra-smooth, mirrow-like surfaces need no grinding or polishing of any kind. As a result, the usual time-consuming and expensive processing of the germanium into usable form is virtually eliminated."

Mosquito Light Trap.—The U. S. Army Engineers have built a better mosquito trap. Developed by their Research and Development Laboratories at Fort Belvoir, Va., the new aluminum trap has proved under test to be more rugged and corrosive-resistant than standard models.

The purpose of the mosquito light trap is to catch a representative sample of a mosquito population to determine its density and species for establishing controls and measuring their effectiveness.

Now standard military equipment, the new trap consists of an aluminum frame, a cover-protected light bulb, a 10 milli-horsepower, electric motor driven fan, a screen cone and a collecting jar. Mosquitoes attracted by the light are blown by the fan down through a screen cone into a plastic jar where they are killed by sodium cyanide or paradichlorobenzene. The trap operates on a 110 volt line, and is controlled by an automatic electric timer.

The device was tested at Fort Stewart, Ga., Jacksonville and Key

West, Fla., and at the Army Chemical Center, Md., as well as at Fort Belvoir.

Electronic Bank Bookkeeping Machine.—An electronic bank bookkeeping machine that could eliminate one of banking's most time-consuming problems—the manual handling of thousands of checking account ledger forms while posting checks and deposits—is currently being demonstrated at regional banking conventions.

Fresh from Burroughs research laboratories, the machine is an electronic bank bookkeeping machine with automatic statement processing. It was developed as a part of the Burroughs MICR (Magnetic Ink Character Recognition) automation program for banks. The machine is being shown at the various conventions to check banker reaction to the equipment and obtain evaluation information.

The device combines the automation talents of Burroughs present electronic bank bookkeeping machines with an automatic statement selector. Upon command from the bookkeeping machine, the selector electronically searches out any desired statement from a stack of statement forms placed in the selector feeding tray and inserts it into the carriage of the bookkeeping machine without human assistance. On completion of posting, it returns the form to a stacker in proper sequence.

Use of the machine could completely eliminate operator handling of the satement forms during posting. Under existing systems operators select statement sheets one by one manually, insert them into the carriage of the bookkeeping machine and after posting remove them from the carriage and place them in a "finished" file.

The manual handling of forms is the most time-consuming—thus, most expensive step in the posting operations, Burroughs officials explained. Elimination of this work through automation could increase both overall speed of the entire posting operation and increase accuracy and efficiency.

With the new Burroughs development an operator would never touch a ledger form while posting. Her work would be reduced to simply reading and posting information from the checks to be posted and indexing this into the machine.

With elimination of manual handling, actual posting speed would be increased to a point where the selector would begin looking for the next statement before posting to the previous account had been completed. By actual count the machine could eliminate 10 posting functions operators now perform manually. In addition, the machine could automatically pass on inactive accounts without loss of time to the operator and still maintain both active and inactive statements in correct amount sequence.

Three magnetic ink stripes now being printed on the reverse side of the statement form and acting as both "memory" and trigger mechanism for the present electronic bookkeeping machine could serve double duty by also acting as the trigger and "memory" for the selector device.

Both the selector and the bookkeeping machine could read information and instructions coded into the magnetic ink stripes to carry out their automatic functions. The information would be "written" into the stripes as a normal by-product of posting operations on the bookkeeping machine.

Carrying automatic functions even further, the machine could also be used to obtain trial balances and transfer balances from old to new statement forms without human aid.